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Aspects and "Grundvorstellungen" of the Concepts of Derivative and Integral

Subject Matter-related Didactical Perspectives of Concept Formation

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Abstract This paper discusses *Aspects* and *"Grundvorstellungen"* in the development of the concepts of derivative and integral, which are considered central to the teaching of calculus in senior high school. We focus on perspectives that are relevant when these concepts are first introduced. In the context of a subject-matter didactical debate, the ideas are separated into two classes: firstly, mathematically motivated aspects, such as the limit of difference quotients or local linearization within the concept of derivative, as well as the product sum, antiderivative, and measure aspects of integration; secondly, the "Grundvorstellungen" associated with the concepts of derivative and integral. We regard finding a comprehensive description of *Aspects* and *"Grundvorstellungen"* an important objective of subject matter didactics. This description should clarify both the differences and the relationships between these perspectives, including mathematically motivated *Aspects* and *"Grundvorstellungen"* that are central to the students' perspective. The primary objectives of this paper include a specification of the concepts of *Aspects*

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and *"Grundvorstellungen"* within the context of differentiation and integration, and a discussion of the relationships between the *Aspects* and *"Grundvorstellungen"* associated with the concepts of derivative and integral. We first present the characteristic properties of *Aspects* and *"Grundvorstellungen"*, including an account of related concepts and the current state of research. *Aspects* and *"Grundvorstellungen"* are analyzed, based on a subject-matter didactical analysis of the concepts of derivative and integral. We conclude with an account of how these insights can be beneficially exploited for introducing differentiation and integration in real-life environments, within the framework of a theory of concept understanding and subject matter didactics.

Keywords Aspects · Grundvorstellungen · Derivative · Integral · Concept

MESC Codes C30 · I40 · I50

Aspekte und Grundvorstellungen zum Ableitungs- und Integralbegriff

Stoffdidaktische Perspektiven zur Begriffsbildung

Zusammenfassung Der Beitrag befasst sich mit *Aspekten* und *Grundvorstellungen* bei der Entwicklung des Ableitungs- und Integralbegriffs, die als zentrale Begriffe des Analysisunterrichts der Sekundarstufe II angesehen werden. Wir konzentrieren uns dabei auf Sichtweisen, die bei der Einführung dieser Begriffe von Bedeutung sind. In der stoffdidaktischen Diskussion werden einerseits verschiedene stärker fachwissenschaftlich fundierte Aspekte wie etwa der Grenzwert des Differenzenquotienten oder die lokale Linearisierung beim Ableitungsbegriff bzw. Produktsummen-, Stammfunktions- oder Maßaspekt beim Integralbegriff unterschieden. Andererseits werden mit dem Ableitungs- und dem Integralbegriff Grundvorstellungen verbunden. Eine wichtige Aufgabe der Stoffdidaktik sehen wir in der umfassenden Beschreibung von *Aspekten* und *Grundvorstellungen* sowie in der Klärung der Unterschiede und Zusammenhänge dieser Sichtweisen, also zwischen den eher fachmathematisch zu begründenden Aspekten sowie den aus der Perspektive der Lernenden zentralen Grundvorstellungen. Die Präzisierung der Begriffe *Aspekt* und *Grundvorstellung* im Kontext der betrachteten Begriffe Ableitung und Integral sowie die Diskussion der Beziehungen zwischen *Aspekten* und *Grundvorstellungen* am Beispiel des Ableitungs- und Integralbegriffs sind zentrale Ziele des Beitrags. Im Beitrag werden zunächst unter Berücksichtigung verwandter Konzepte und des aktuellen Forschungsstandes die charakteristischen Eigenschaften von *Aspekten* und *Grundvorstellungen* vorgestellt. Dann werden diese auf der Basis einer stoffdidaktischen Analyse für den Ableitungs- und Integralbegriff analysiert. Abschließend geben wir einen Ausblick, wie diese Erkenntnisse unter den aktuellen Rahmenbedingungen im Rahmen von Begriffsverständnis und Stoffdidaktik zur Einführung des Ableitungs- und Integralbegriffs gewinnbringend genutzt werden können.

Schlüsselwörter Aspekte · Grundvorstellungen · Ableitung · Integral · Begriff

1 Aspects and Grundvorstellungen of Mathematical Concepts

According to Vollrath (1984) and Weigand (2014), in order to understand a mathematical concept, students need to acquire knowledge of some features or properties of the concept, together with the relations between them – the *concept content*. They need to acquire an overview of all objects that are subsumed under that concept – the *concept scope.* They should be aware of relations between this concept and others – the *concept network*. Students should also acquire some knowledge of concept *applications* and the *ability to manipulate* the concept. The aim of mathematics lessons is to provide students with a comprehensive understanding of central mathematical concepts in the sense outlined here. The teacher's role is to plan, initiate, support, guide, and monitor the process of concept formation.

With respect to the concepts of calculus, there are numerous didactic ideas, suggested teaching methods, empirical studies, and practical investigations within the framework of curricula and mathematical textbooks (e. g. Rasmussen and Borba 2014). Nevertheless, Rasmussen et al. (2014) came to the following conclusion:

> While the past several decades of research in calculus has [sic] contributed to better understanding of mathematical thinking, learning, and teaching in areas such as limit, derivative, and integral, too much research remains isolated and uncoordinated. (ibid., p. 508)

The present article aims at providing a foundation for overcoming this problem. To this effect, mathematical aspects of both differentiation and integration are identified, together with associated Grundvorstellungen ("Grundvorstellungen" could be translated with "basic mental models") (Hofe 1995; Blum et al. 2004, 2005), and the relationships between them. This structures the complex subject matter of derivatives and integrals, enabling these two central concepts in calculus to be considered and studied from multiple perspectives with respect to mathematics education.

1.1 The Concepts of "Aspects" and "Grundvorstellungen"

Students' *Grundvorstellungen* of mathematical concepts have been discussed in German-language pedagogy and the didactics of mathematics for more than 200 years, for example, by Pestalozzi, Herbart, Kühnel, Breidenbach, Oehl, and Griesel (cf. Hofe 1995, p. 22). Grundvorstellungen give meaning to content-based *aspects* of a mathematical concept, providing relations to meaningful contexts. This is a crucial prerequisite for working meaningfully with a concept. We define two expressions that are basic for the present article:

> An *Aspect* of a mathematical concept is a subdomain of the concept that can be used to characterize it on the basis of mathematical content.

> A *Grundvorstellung* of a mathematical concept is a conceptual interpretation that gives it meaning.

Kilpatrick et al. (2005) established the need to study the concept of "meaning," arguing that it influences the process of learning mathematics. For them, the meaning

of a concept is constituted firstly by its mathematical significance, which should not be separated from its genesis: "To understand the meaning of a concept, theorem or mathematical idea, it is important to appreciate the process through which this entity has evolved. ... Thus, the constructive processes, rather than the referential elements, provide meaning." (ibid., p. 2). Secondly, a subjective or individual viewpoint underlies the process of constructing meaning. This is the interpretation of the concept for the individual, the integration of the concept into the individual's personal worldview (cf., e. g. Koller 2008).

The relation *"Aspect–Grundvorstellung"* of a given mathematical concept is not one-to-one. An aspect of a mathematical concept can be the basis of several Grundvorstellungen. Vice versa, a specific Grundvorstellung can be developed with respect to several aspects and give them meaning. We will discuss this relation in detail for the concepts "derivative" and "definite integral" in Sects. 3 and 4 (see also Figs. 4 and 5).

1.2 Grundvorstellungen as a Prescriptive and a Descriptive Notion

The concept of Grundvorstellungen can be used in both a prescriptive and a descriptive sense (cf. e. g. Hofe et al. 2005):

Grundvorstellungen as a *prescriptive* notion are the answer to the following subject-matter didactic question: How should students generally and ideally think of a given mathematical concept? These Grundvorstellungen result from a subject-matter didactical analysis of the mathematical concept in question. Supporting students in developing these Grundvorstellungen is one of the objectives of mathematical teaching. Thus, they provide teachers with guidance for organizing lessons.

By contrast, Grundvorstellungen as a *descriptive* notion answer the subject-matter didactic question of how a given student actually thinks about a given mathematical concept. Grundvorstellungen in this sense are the result of individual learning processes. By observing students as they work, and analyzing the oral and written output, teachers can attempt to gain insight into students' individual Grundvorstellungen. These can then serve as a starting point for teaching and support activities within a given learning group.

When referring to "Grundvorstellungen" in Sects. 2–5, we always refer to Grundvorstellungen in a prescriptive sense. Through a subject-matter didactic analysis of the concepts of derivative and integral, central aspects and associated Grundvorstellungen will be established, and relations between these two viewpoints highlighted.

1.3 Relation to the Idea "Concept Image–Concept Definition"

Grundvorstellungen of mathematical concepts can also be considered within the theoretical framework of "concept image–concept definition." These terms have been used in the didactics of mathematics since the early 1980s to distinguish between technical issues of a concept and the associated mental images (e. g. Vinner and Hershkowitz 1980; Tall and Vinner 1981; Bingolbali and Monaghan 2008, p. 31). "Concept definition" refers to the formal or explicit definition of a particular concept. Tall and Vinner (1981) regard it "to be a form of words used to specify that concept"

(ibid., p. 152). "Concept image" refers to all individual mental images identified with the concept. This term describes "the total cognitive structure that is associated with the concept [...]. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures." (ibid. p. 152)

The relation *concept image–concept definition* has been studied in many different contexts, such as "irrational numbers" (Sirotic and Zazkis 2007, p. 49) or "limits of sequences" (Roh 2008, p. 218). A recurring problem addressed in these publications is that the concept image associated with a given concept definition is very narrow. In addition, students are in danger of drawing conclusions about the concept definition by generalizing a concept image that focuses exclusively on certain special cases only (Vinner 2011, p. 248). This danger is particularly present in the basic concepts of calculus, given that in specific classroom environments and especially in exam assignments, there is a bias toward calculation-oriented exercises, which are easily practiced beforehand on a formal, symbolic level. Törner et al. (2014, p. 547) remark that: "However, in some nations, teaching ... calculus in the classroom is rather traditional, focusing on procedural aspects of knowledge."

The relations *concept image–Grundvorstellung* and *concept definition–Aspect* can be described as follows: A concept image may contain several individual Grundvorstellungen that conceptualize different perspectives of that concept. Individual Grundvorstellungen are central components of a valid concept image (for a detailed analysis, cf. Rembowski in press). These Grundvorstellungen give meaning to mathematical concepts that may be studied with respect to various aspects (see Sect. 1.1). Each of these aspects may be expressed with one of the various concept definitions that one reads in textbooks. Thus, a concept definition is a specific realization of an aspect. These linkages are shown in Fig. 1, which illustrates that the relation between concept definition and concept image is highly nontrivial. This relation may become more tangible by taking Grundvorstellungen and aspects and their relations into account.

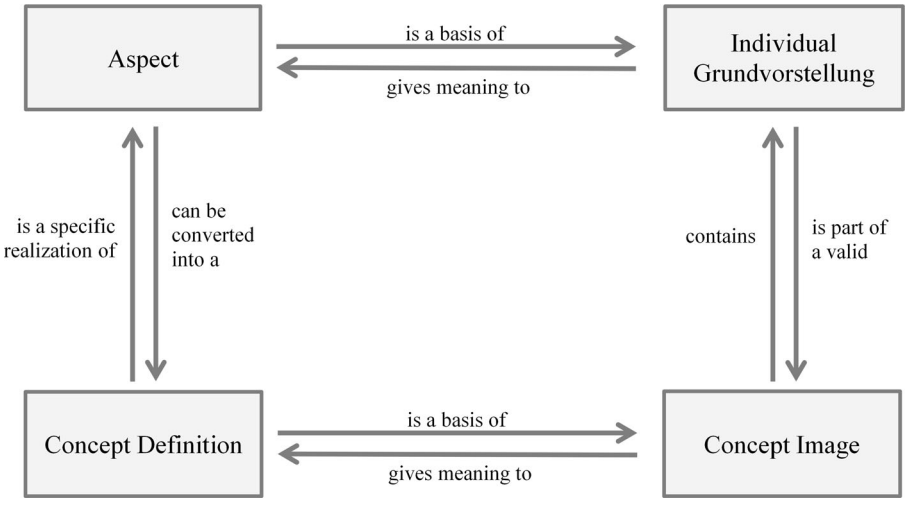


Fig. 1 Relations between aspect, Grundvorstellung, concept definition, and concept image

2 The Limit Concept

The concept of limit is a basic concept in calculus and may even be the most important basis of the concepts of derivative and integral. Up to the 1960s, calculus lessons at German high schools had been developed in close association with university mathematics. The sequence concept served as a basis for the limit concept and therefore, the "ϵ-n_0 definition" for the limit of a sequence. This strict approach to the limit concept, in relation to the sequence concept, had been widely criticized (cf. Pickert 1962) and some different alternatives concerning the approach to calculus in high schools were developed. These are the linear approximation concept, the "Lipschitz analysis," and especially the "intuitive limit concept," which is/are based on the college lessons of Serge Lang (1927–2005) and Emil Artin (1898–1962; cf. Weigand 1993).

2.1 The Intuitive Limit Concept

Nowadays, the formal definition of the limit is of only minor importance in German high school curricula. The German learning standards (KMK 2012) emphasize the conception of the "intuitive limit concept." It is stated that students should use "limit concepts on the basis of a preliminary limit concept, especially for the determination of differentiation and integration" (p. 22). The notion of a sequence no longer appears in these standards. In many curricula of the German federal states, its significance has been reduced considerably, or it has even been entirely removed from mathematics classrooms.

Following the historical development of the limit concept, starting from the mathematics of the ancient Greeks, and extending to Newton, Leibniz, Cauchy, and Weierstraß to the current version of an "ϵ-n_0" or "ϵ-δ definition," there are two main aspects related to this concept: the *dynamic aspect* and the *static aspect*. The former is closely related to the possibility of "and so on" or a perception of a potentially infinite process based on the (mental) successive pacing through (the beginning of) the natural number sequence. The dynamic aspect may lead to "infinitely great numbers" or an "infinitely close approach" to a fixed value. Looking at the limit concept under a *static aspect* means regarding a sequence $n \to a_n$, $n \in \mathbb{N}$, $a_n \in \mathbb{R}$, and in contrast to the mental dynamic process of moving along the (natural) number line versus infinity, now considering the "reversal" of the argumentation. Starting with a fixed value or object L and a given value $\epsilon > 0, \epsilon \in \mathbb{R}$, we are looking for a special (natural) number $n_0 \in \mathbb{N}$, considering all values n with $n > n_0$ and values a_n closer to L than ϵ.

The intuitive limit concept builds on dynamic ideas, with alterable variables or values seen as "fluent" or successively discretely changing. Especially the gradual discrete approach to "infinity" is, in terms of "walking along" the natural number line, an intuitive and basic concept and a key term in the development of the limit concept.

Based on the *dynamic* and *static* aspect, two different Grundvorstellungen can be formulated:

2.2 The Grundvorstellung "Approach"

The pursuit of numerical values versus infinity and any approach of a fixed value in terms of reconstructing actions can be well described by stepwise – discrete – approximations. Thus, the reasoning is closely related to perceptions connected with the sequence concept. Perceiving the approach of the sequence values to a fixed value, while the "n-values" are coverging towards infinity, is the basic intuitive perception of a limit. The – potential – possibility of any continuation is the intuitive perception of an infinite process.

In mathematics teaching, the development of this perception starts with explicitly given, convergent sequences and continues with iteration or recursively defined sequences. It is then specified with interval nesting processes and the definition of a tangent as the limit of a family of secants. Students have to develop the following competencies in particular. They must:

- Recognize the (in principle) arbitrary continuation of the natural numbers, which is the basis for a dynamic "infinity process"
- Recognize the unrestrained continuation of an "infinite process," which is the possibility of coming arbitrarily close to a "limit object"
- Be familiar with specific examples of "infinite processes" and appropriate "limit objects" at the symbolic, iconic and enactive levels
- Combine graphical and numerical ideas with the process of "pursue to" or "is arbitrarily close to"
- Explain the concepts of derivative and integral in the sense of the approach perception.

2.3 The Grundvorstellung "Neighborhood"

The dynamic process of continuation in the sense of pursuing a sequence terminates with the reverse way of thinking. Starting with a given size of neighborhood of a fixed (limit) value, we are now looking for a special n_0-value of the sequence whose associated sequence value satisfies a certain condition. The Grundvorstellung "neighborhood" is based on the idea that all values of a sequence (or function) – starting with a special n_0-value – are located in even the smallest environment of the limit value.

The students:

- Connect graphical and numerical perceptions with the concept "environment" and see the relations to the values of a sequence or function
- Can *verbally describe* the idea of the limit definition (in the sense of the "ϵ-n_0 definition")
- Can numerically and graphically describe the limit process with recursively defined sequences and know criteria for a convergent behavior of sequences with $a_{k+1} = A \cdot a_k + B$, $A, B \in \mathbb{R}$, $A = const$, $B = const$
- Can describe interval nesting processes and the approximately iterative solving of equations with the idea of a limit process.

In an educational process that is based on understanding the central mathematical basics of calculus, the aim is to develop aspects, basic ideas, and Grundvorstellungen of calculus in a way that makes it possible to proceed to strongly formal notions. Törner et al. (2014) also emphasize this in their metastudy on calculus lessons in European countries: "From the literature review, the progression from informal to formal knowledge seems to be a goal for calculus teaching" (p. 558). But this is – with the present state of school mathematics – only possible in advanced calculus courses.

3 Aspects and Grundvorstellungen of the Concept of Derivative

Following the method of subject matter didactics ("Stoffdidaktik"), we start by analyzing the mathematical definition of a derivative. Differentiation can be defined in various ways and we elaborate on this by defining two aspects of this concept in the first subsection. The second subsection makes the transition from these aspects to the notions that students should develop.

3.1 The Aspect "Limit of Difference Quotient"

When considering the aspect of differentiation as the limit of a difference quotient, the first step is to analyze the difference quotient $\frac{f(x)-f(x_0)}{x-x_0}$. In certain situations, it is not this quotient but rather its limit for the two points x, x_0 converging. This yields the limit value that is used in the definition of the differential quotient.

3.1.1 Differentiation as the Limit of a Quotient of Differences

Let f be a real-valued function defined in some neighborhood of the point $x_0 \in \mathbb{R}$. We say that f is differentiable at the point x_0 if the limit $f'(x_0) := \lim\limits_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ (or equivalently $f'(x_0) := \lim\limits_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$) exists. We refer to this limit as the derivative of f at x_0.

Besides the small variation between the two equivalent ways of writing the limit given in this definition, there are also other versions too. For example, one may say that $f'(x_0)$ is the unique value that removes the singularity 0 from the function $h \longmapsto \frac{f(x_0+h)-f(x_0)}{h}$. We consider this to be another formulation of the same aspect, because the notion of removing a singularity is defined in terms of limits and the same expression is involved.

3.2 The Aspect "Local Linearization"

Another aspect stems from the fundamental idea of approximation; we attempt to approximate the values of the function by a linear function in an (arbitrarily) small neighborhood of x_0. If the function is indeed linear, then any function value $f(x)$ can be computed from the $f(x_0)$ value at some fixed x_0 by adding a multiple of

the distance $x - x_0$. The multiplication factor is the slope $m \in \mathbb{R}$; thus we have $f(x) = f(x_0) + m \cdot (x - x_0)$. If the function is not really linear, one could try to approximate it linearly. To do this, we modify the slope by an additive correction function: $f(x) = f(x_0) + (m + \delta(x - x_0)) \cdot (x - x_0)$. The linear part is a good approximation locally, if the correction vanishes near x_0.

3.2.1 Differentiation as Local Linearization

Let f be a real-valued function defined in some neighborhood of the point $x_0 \in \mathbb{R}$. We say that f is differentiable at the point x_0 if there is some number m such that

$$f(x) = f(x_0) + (m + \delta(x - x_0)) \cdot (x - x_0),$$

where the slope correction $\delta(x - x_0)$ is such that: $\lim\limits_{x \to x_0} \delta(x - x_0) = 0$. We call this number m the derivative $f'(x_0)$.

Equivalently, one can define $r(x - x_0) := (x - x_0) \cdot \delta(x - x_0)$ that satisfies $\lim\limits_{x \to x_0} \frac{r(x - x_0)}{x - x_0} = 0$ and yields the linear approximation in the form $f(x) = f(x_0) + m \cdot (x - x_0) + r(x - x_0)$.

We also consider differentials to be a means of expressing this aspect, and not another independent aspect.

The approaches in these two definitions (two aspects) differ, and different mathematical operations are performed in order to arrive at the concept of a derivative. For example, local linearization allows for a generalization to higher dimensions, and to potentially simpler proofs of the rules for calculating derivatives. On the other hand, approaching the concept via the limit value of difference quotients has practical advantages, in particular for calculating the derivative of simple power functions.

These two definitions are mathematically equivalent, but they reveal different fundamental properties. They shed light on two aspects of the concept of derivative. From a didactic perspective, the fact that both aspects of differentiation correspond to different concept images for students is important. The aspect of differentiation as the limit value of a difference quotient supports concepts of speed and rates of change.

The interpretation of differentiation as a local linear approximation helps students to understand the error between the optimally approximating linear function and the original function, and about the possibility of describing the function as linear in a small neighborhood – its graph appears as a straight line when zoomed in at a particular point (function microscope, cf. Kirsch 1979).

This completes the discussion of our second and last aspect. Before we move on to Grundvorstellungen, it should be noted that there are other possible aspects that we ignore. For example another aspect may be drawn from nonstandard analysis, where the derivative is a quotient of infinitesimals.

3.3 The Grundvorstellung "Local Rate of Change"

In the lower grades of secondary education, various concepts describing processes of change are introduced, including absolute and relative variation, percentage change, and monotonicity. This prompts the question of how fast a variation can occur as a function of an argument, consequently leading to the idea of considering the rate over an interval (cf. Hahn and Prediger 2008). For a dependent entity *f,* the absolute change in value over the interval $[x, x_0]$ as a function of the argument value is $f(x) - f(x_0)$ and its relative change, or synonymously, rate of change, is described by the expression $\frac{f(x)-f(x_0)}{x-x_0}$ – the difference quotient. Hence, rate of change is an interpretation of the difference quotient (one may use Frege's language to say that "rate of change" is the "sense" and "difference quotient" the "meaning" of this concept). The rate of change over an interval averages the change over this interval. Considering a sequence of intervals that tend to a number, one arrives at the notion of local rate of change, i. e., the differential quotient.

Local rates of change are important in many areas of applied mathematics and, hence, there are many ways of motivating ideas of calculus from these domains.

Studies (e. g. Herbert and Pierce 2012) have shown that the concepts of rate or rate of change are complex, yielding multiple perspectives (e. g. Thompson 1994; Ubuz 2007), and "[...] students at various levels have difficulty conceptualizing the idea of a rate of change" (cf. Teuscher and Reys 2010, p. 519). Herbert and Pierce (2012) demonstrate that for some students, the rate of change is a single object, whereas for others, it is composed of two distinct changes. For some, their concept image is primarily associated with the expression used in calculations. Yet other students assume that rates of change are always constant – possibly due to generalizing from linear graphs (cf. Herbert and Pierce 2012, p. 94). Thompson and Thompson (1994, 1996) show that teaching the Grundvorstellung "local rate of change" places substantial demands on teachers.

The local (or instantaneous) rate of change can be obtained as the limit of the difference quotient (first aspect). The interpretation of the content of this aspect (particularly by means of dependent entities) when the differences are regarded as changes, is the Grundvorstellung "local rate of change."

A comprehensive, explicit Grundvorstellung of the local rate of change includes:

- The conception of the instantaneous speed of a process (the conception of instantaneous velocity can be used as a prototype)
- The conception that the change in a dependent entity is given by $\Delta y \approx f'(x) \cdot \Delta x$

Using technology, it may be useful to give students the opportunity to experience rates of change in virtual worlds and this may have positive effects (Thompson et al. 2013).

If the equation $\Delta y \approx f'(x) \cdot \Delta x$ is considered algebraically, and reformulated, one immediately obtains an approximation of the derivative $\frac{\Delta y}{\Delta x} \approx f'(x)$. In particular, when reading off a slope value (from slope triangles), the representation $\frac{\Delta y}{\Delta x}$ is used.

This shows that local rates of change are not the only Grundvorstellung that need to be discussed; there are many other possibilities.

3.4 The Grundvorstellung "Tangent Slope"

Students become acquainted with the concept of slope during secondary education (Crawford and Scott 2000), which is associated with an extension of the concept of tangent (cf. Friedrich 2001). Using what is known as the "touch point property" ("Berührpunkteigenschaft," cf. Möller 2013, p. 20; Tall 2012, p. 307), the concept image that the tangent of a graph does not re-intersect with the graph is established (cf. Büchter 2014, p. 45) – as shown in Fig. 2. Büchter (2012, p. 171) explains this approach: "In [...] school textbooks, concept development for circle tangents is essentially reduced to the property of having 'exactly one common point'." Conceptual change (cf. Tirosh and Tsamir 2004; Vosniadou and Verschaffel 2004) is necessary to convert this into a conception that is viable in calculus. Interestingly (but not surprisingly), physical experiences are rather close to the calculus concept of tangent: When a body is guided along a curve and the guiding force vanishes, it will move on along a straight line that is the tangent to the former trajectory (e. g. sparks flying from an angle grinder).

This interpretation has historical roots. Even in ancient Greek mathematics, in Book III of Euclid's *Elements* (cf. Deak 2010), an analogous definition of the concept of tangent can be found: "We say that a straight line touches a circle (is tangent to that circle) if it encounters the circle, but its extension does not." If this concept image is extended to arbitrary curves without critical thinking, concept

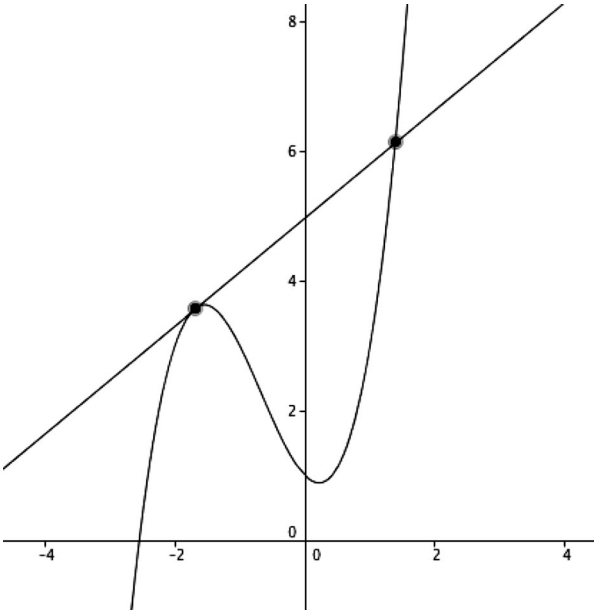


Fig. 2 Tangent to a curve

development becomes incompatible with those of calculus. This concept image of tangents must be modified (Stump 1999), in particular so that it does not contradict the first definition of the differential expression.

On the other hand, the interpretation of tangents as locally tangential lines (cf. Blum and Törner 1983, p. 93; Danckwerts and Vogel 2006) is appropriate and consistent. In this case, tangents are understood as lines that are *locally* tangential to the graph (Blum and Törner 1983; Biza 2011; Tall 1987). In order to judge whether a line is a tangent, it is sufficient to look (graphically in the verbal sense, algebraically in a metaphorical sense) at an arbitrarily small interval around the point in question. Whatever may happen outside this interval is not relevant to answering this question. This local viewpoint is a key idea of calculus that is new to students.

Here, an important difference occurs between the concept images of tangents, as discussed by Vinner (1991), and the Grundvorstellung of a tangent. We discuss the Grundvorstellung "derivative as the slope of the tangent," so that the tangent is subordinate to the derivative. On the other hand, Vinner's concept image is more self-contained: "The concept image is something non-verbal associated in our mind with the concept name. It can be a visual representation [or] a collection of impressions or experiences" (Vinner 1991, p. 87). Thus, he allows the line with equation $x = 0$ to be seen as tangent to the graph of $\mathbb{R} \ni x \longmapsto \sqrt{|x|}$, which is not part of our Grundvorstellung.

The connection between this Grundvorstellung and the associated aspects is quite intricate. From the point of view of the developed mathematics, the derivative is the primary object and the tangent is defined only in terms of it. From the genetic perspective of how mathematics develops, however, one may say that the tangent is an intuitive concept that can be modelled by either of the aspects described earlier.

Understanding the derivative as the slope of the tangent has the positive effect that one can easily understand why the slope of $x \longmapsto k \cdot f(x)$ and $x \longmapsto f(k \cdot x)$ is k times higher than that of f. That is, a slope triangle attached to the tangent gets "stretched" (or compressed) in the y-direction or x-direction, resulting in an amplification of the slope by the factor. Similar geometric arguments clarify the situation for $x \longmapsto a + f(x + b)$. As the graph of the inverse function is the mirror image with respect to the line $y = x$, one can, moreover, imagine the tangent and a slope triangle mirrored as well, in order to understand the rule for the derivative of the inverse function.

This Grundvorstellung includes the conception:

- tangents as locally tangential lines
- the slope of a line at a point
- a tangent gives the local direction of a curve

3.5 The Grundvorstellung "Local Linearity"

Curves can be approximated by piecewise-linear curves (Blum and Törner 1983, p. 96). The idea of thinking about smooth curves as locally linear is first introduced in lower secondary education (e. g. using polygons to approximate circles and other

curves, cf. Oldenburg 2012), and can further be developed into a Grundvorstellung of differentiability and of derivatives.

We consider an arbitrary function f at a point x_0 and seek a linear function that locally approximates the behavior of the function as accurately as possible (cf. Teague 1996).

There are infinitely many lines that pass through the point $(x_0, f(x_0))$ on the graph (cf. Danckwerts and Vogel 2006, p. 68). Each has a characteristic slope m. The equation of each line is given by $g(x) = f(x_0) + (x - x_0) \cdot m$. The difference in function value between f and g near the point x_0 is $r(h) = f(x_0 + h) - g(x_0 + h) = f(x_0 + h) - f(x_0) - h \cdot m$ and by continuity this error approaches zero as $x \to x_0$ for every value of m, but for one specific value, even the relative approximation error satisfies:

$$\lim_{h \to 0} \frac{r(h)}{h} = 0$$

Thus, the line of best approximation has a slope equal to exactly $f'(x_0)$, the derivative of f, that can be understood as the slope of the tangent. This gives an example of a "multiple-linked representation" (cf. Tall 1991, p. 33), which "allows a person to use several different representations at the same time, switching from one to another when it is appropriate to do so" (cf. Mackie 2002).

The basic idea is to observe the graph with a (metaphorical) microscope, and recognize the straight-line behavior that is present in differentiable functions. To this end, *dx* and *dy* can be defined as the components of a vector in the corresponding direction. Studies performed by Tall (2012, p. 289 ff.) show that students taught from this perspective develop more sound conceptions of tangents. The idea that the graph of a differentiable function is locally linear corresponds with the idea that a function that is differentiable in all points of an interval can be approximated by a linear spline. The idea of building up curved graphs from simple (linear) but very small components dates back to the inventors of calculus (Jahnke 1999, pp. 89–129).

A comprehensive, explicit Grundvorstellung of local linearity includes:

- When zooming in very close to a point of the graph of a differentiable function, one sees an almost straight segment.
- For small changes in argument, the function is essentially linear, so that it can be approximated with a linear model.

We demonstrate how this Grundvorstellung promotes understanding the structure of the chain rule $(u(v(x)))' = u'(v(x)) \cdot v'(x)$. Where does the product come from in the expression? The functions are approximated locally by linear functions, the composition of which yields a function whose slope is the product of those of the original two functions.

Moreover, it is clear that at local extrema, the derivative must vanish, as otherwise one would have a direction in which the function values will decrease or increase.

3.6 The Grundvorstellung "Amplification Factor"

If there is a functional relationship between (two) parameters, changes (or uncertainty, such as errors in measurement) in the independent parameter induce changes in the dependent parameter. The Grundvorstellung of amplification factors for small changes is that they are proportional.

The Grundvorstellung of amplification factors can be used both for the difference quotient and the differential quotient (cf. also Malle 2003). This can help to bridge the transition and is in a certain sense (as evident in the second definition), closely related to the idea of local linearization.

Most current calculus textbooks do not place any emphasis on this Grundvorstellung, although it is very useful. For example, it explains how differentiation can work similarly to a "variation detector." Wherever a function has little variation, the values of the derivative are close to zero, and wherever the changes are significant, the derivative is large. An example of this is given in Fig. 3.

Mamolo and Zazkis (2012) record that students experience difficulties with exercises that require this Grundvorstellung. Although they do not explicitly use the notion of an amplification factor, their work may indicate that the students they observed had an insufficiently developed Grundvorstellung of differentiation as an amplification factor.

A comprehensive, explicit Grundvorstellung of amplitude factors includes the following components:

- Differentiation measures the extent of changes in the dependent parameter induced by small changes in the independent parameter.
- High values of the derivative indicate fast/large variation.
- For small changes, Δx and Δy are approximately directly proportional.

This Grundvorstellung enables an understanding of the structure of the chain rule $(u(v(x)))' = u'(v(x)) \cdot v'(x)$ in the following way: When two variational processes are combined, each multiplies the change by its own amplitude factor, so that overall, factors are multiplied.

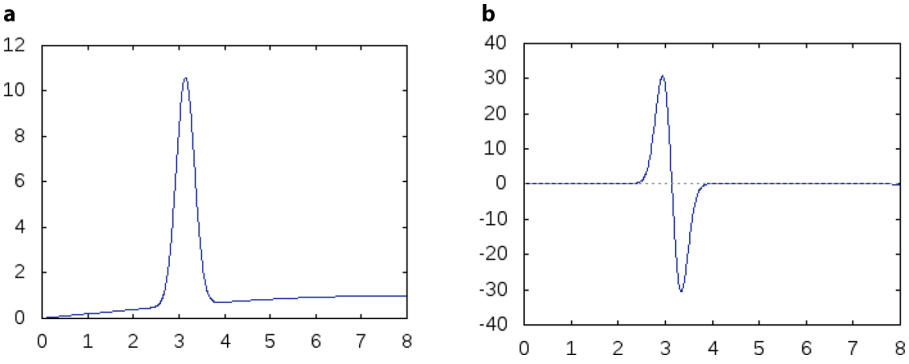


Fig. 3 **a** Slowly varying function with a strong variation near one point x. **b** Its derivative, showing the large effect of small changes near x

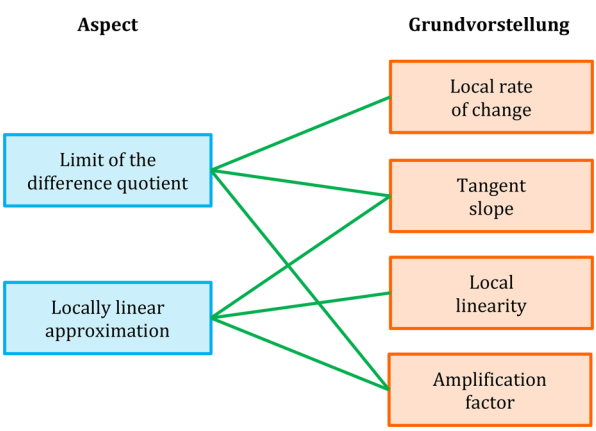


Fig. 4 Aspects and Grundvorstellungen of differentiation

3.7 Overview: Aspects and Grundvorstellungen of the Concept of Derivative

From a subject-matter didactic analysis of the concept of derivative, two aspects and four Grundvorstellungen were identified. These are summarized in Fig. 4, including the relations between the different aspects and Grundvorstellungen. The connecting lines indicate that the aspect is a basis of the related Grundvorstellung and that the Grundvorstellung gives meaning to the aspect (see Sect. 1.4). For example, the aspect that the derivative is a limit of a difference quotient may be understood either by interpreting it as the limit of average change rates, as the limit of secant slopes, or as amplification factor. On the other hand, there is no linking line from this aspect to "local linearity," as this Grundvorstellung cannot really be built up from a definition that emphasizes this aspect.

In the preceding analysis, we have given several small examples that show how Grundvorstellungen may help students to understand calculus. However, note that each Grundvorstellung has particular limitations and there are situations in which none is sufficient to yield a definitive conclusion. Consider, for example, the function with $f(0) = 0$, $f(x) = x^2 \sin(\frac{1}{x})$ for $x \neq 0$. None of the Grundvorstellungen is precise enough to reveal whether this function is differentiable at the origin (which is the case). However, the Grundvorstellungen may render the property of differentiability plausible; one can zoom into the graph and have the visual sensation of the graph becoming straight, or one may note that the sine function is bounded by 1, so that small changes of x cannot be amplified by more than the amplification of the quadratic function – and thus one expects the function to be differentiable.

4 Aspects and Grundvorstellungen of the Concept of Definite Integral

In this section, important aspects and Grundvorstellungen associated with the concept of a definite integral are discussed. Because aspects and Grundvorstellungen play an important role in introducing of the concept of integral, we focus here on the definite integral and not on the integral $I_a(x) = \int_a^x f(t)\,dt$ as a function. The integral as a function becomes more important at a later time in mathematics teaching.

Firstly, we distinguish between the aspect of a definite integral as a product sum, the aspect of anti-derivatives, and the aspect of measure. The defining property of these aspects is that they can all be used to define or technically characterize the concept of a definite integral. Secondly, we present the Grundvorstellungen of (re)construction, area, average value, and accumulation. Finally, the relations between the different aspects and the different Grundvorstellungen will be discussed, as well as alternative perspectives of the concept of definite integral.

4.1 The "Product Sum" Aspect

By a product sum, we mean an expression of the type: $a_1 \cdot b_1 + a_2 \cdot b_2 + ... + a_n \cdot b_n$. The Riemann integral can be defined and calculated using product sums. To define the definite integral, we consider a function f defined and bounded on a closed real interval $[a, b]$. The interval is split into n subintervals $[t_{i-1}, t_i]$, where $t_0, t_1, ..., t_n$ are finitely many points such that $a = t_0 < t_1 < t_2 < ... < t_n = b$. Within each subinterval, the supremum and infimum of f are both finite, as f is bounded. Let the supremum of f on the subinterval $[t_{i-1}, t_i]$ be denoted M_i, and the infimum be denoted m_i. $Z = [t_0 = a, t_1, t_2, ..., t_n = b](t_i \in \mathbb{R}, 1 \leq i \leq n, n \in \mathbb{N})$ is then a partition of the interval $[a, b]$. The product sum $\sum_{i=1}^{n} M_i \cdot (t_i - t_{i-1})$ is the *upper sum*, and the product sum $\sum_{i=1}^{n} m_i \cdot (t_i - t_{i-1})$ is the *lower sum* of f for the partition Z. If, for a given function f, bounded on the closed interval $[a, b]$, the supremum S of all lower sums is equal to the infimum I of all upper sums, then this number is defined to be the *Riemann integral of f on the interval* $[a, b]$ (cf. Walter 2004, p. 197 ff.).

In addition to upper and lower sums, it is also possible to consider intermediate sums, which are also product sums. In many applications in which definite integrals are calculated, the product sums of two distinct entities are considered, yielding a third entity. A possible example is the product sum comprising time and velocity, which yields the physical entity of distance. Whenever the definite integral is interpreted as the limit of upper, lower or intermediate sums, or if specific product sums are calculated from given numbers, the emphasis is on the product sum aspect of integration.

There are two possible interpretations of definite integrals when considering product sums. As product sums involve both addition and multiplication, one can choose which of the two operations to emphasize. The first interpretation is the idea of a generalized sum, which belongs to the Grundvorstellung of accumulation. But the definite integral can also be thought of as a generalized product. This focuses more on the result of the calculation; as in the case of physical work, the definite integral has units of the corresponding product of force and distance. Blum and Törner (1983, p. 158 ff.) also describe these two viewpoints of product sums, i. e., *integration as generalized summation*, and *integration as generalized multiplication (of physical quantities)*, as basic ways of understanding the definite integral concept.

4.2 The "Anti-Derivative" Aspect

Definite integrals can also be defined using the concept of anti-derivatives. Given a function $f : I \longrightarrow \mathbb{R}$ defined on an interval I, we say that $F : I \longrightarrow \mathbb{R}$ is an *anti-de*

rivative of f if F is differentiable and $F' = f$. This form of inverse differentiation makes it possible to find anti-derivatives of elementary functions, e. g. power functions, polynomials, trigonometric functions, and exponential functions. It is then possible to *define* a certain class of definite integrals in the following way:

Let $f : I \longrightarrow \mathbb{R}$ be a function defined on an interval I with anti-derivative F. For $a, b \in I$, we define: $\int_a^b f(x)\,dx := F(b) - F(a)$

It remains to be determined whether that this definition is well-defined, i. e., that the value does not depend on the choice of anti-derivative. However, if G is another anti-derivative of f, then F and G can only differ by an additive constant, as $(F - G)' = F' - G' = 0$.

This approach leads readily to a method for calculating certain types of definite integral. By defining definite integrals as anti-derivatives, it becomes clear that the operations of differentiation and integration are mutually opposing. Anti-derivatives can then be interpreted meaningfully as a "revision of differentiation" (Bender 1990a, p. 75) and used "as a base for constructing the *function matching* symbolic form" (Jones 2013, p. 138). This aspect becomes even clearer when treating the integral with a variable upper limit as a function. However, in this context, we only consider the definite integral.

With the aforementioned definition of the definite integral, the fundamental theorem of calculus is used as a definition to link together the concepts of derivative and definite integral. The assumption of function continuity that is a necessary hypothesis in the theorem is somewhat obscured. In addition, questions of the existence of definite integrals cannot be adequately discussed within the framework given by a definition of this type, because "Riemann integrability" and "having an anti-derivative" are nonequivalent properties of functions. It is possible to give examples of functions that possess anti-derivatives and yet are not integrable, and conversely, there are functions that are integrable but do not have an anti-derivative.

When using the "concept of anti-derivative and its use for calculating definite integrals" (Kouropatov and Dreyfus 2013, p. 646), which is based on the fundamental theorem of calculus, instead of the aforementioned definition, the issues described previously do not arise. As a result, the aspect could be introduced later on in the course and, moreover, analytically connect some unrelated topics like the gradient of tangents and the area below curves.

4.3 The "Measure" Aspect

Definite integrals are used for measuring length, area, and volume in the context of their measure aspect, and are interpreted as a measure for certain representatives of these physical quantities. The mathematical meaning of measure is an ordering of suitable sets of real numbers, which – depending on the nature of the set – can be interpreted as lengths, areas, or volumes. In particular, measure is non-negative, monotonous, and additive. The definite integral also satisfies these fundamental properties of measure whenever it is applied for measuring areas, lengths (e. g. curve lengths), and volumes (e. g. bodies of rotation). It should be noted that the value of the definite integral is not inherently non-negative, but only acquires this

property when used in a certain way, such as measuring the area above the x-axis. Interpreted in this way, the definite integral is a measure of areas, lengths, and volumes. This perspective is a generalization of the area perception of integration, which includes lengths, volumes, and potentially higher-dimension quantities.

Measure theory and the Lebesgue integral can be considered as the mathematical foundation of the measure aspect of integration. This concept of definite integral allows functions to be integrated over arbitrary measure spaces. It is a generalization of the Riemann integral, i. e., there are Lebesgue integrable functions that are not Riemann integrable. However, the measure aspect should not imply a focus on the Lebesgue integral, as it is also possible to calculate length, areas, and volumes with the help of the Riemann integral.

The aspects described earlier provide interpretations of three separate subthemes of the concept of definite integral. Whereas the product sum aspect primarily emphasizes developing the concept of a definite integral from the Riemannian approach, the anti-derivative aspect highlights the fundamental theorem of calculus and thus underlines the link between integration and differentiation. The measure aspect, on the other hand, illustrates the application of integration in making measurements, and the link to the Lebesgue integral. Narrowing down viewpoints to one or more specific aspects of integration is generally seen as questionable (Huang 2012, p. 167).

In the next section, we will describe the Grundvorstellungen associated with the aforementioned aspects, which can be understood as meaning-constructing interpretations of the definite integral concept.

4.4 The Grundvorstellung "Area"

The Grundvorstellung of definite integrals as area emphasizes one of the applications of definite integrals within the framework of its measure aspect. Whereas area is always non-negative, measuring areas with integration naively, yields the *oriented area*, i. e., areas under the x-axis contribute negatively. Thus, it is the net area that is measured. The actual area can be measured by considering multiple subregions between points of intersection with the x-axis or with the graph of another function. The interpretation of definite integrals as an area is a Grundvorstellung and not an aspect, as measuring area links the definite integral concept with physical experiences from everyday life ("the action of measuring area has and always will have explicit real-life elements to it ..." Bender 1991, p. 51).

Example: Approximating an Area
Approximate the area between the function of f with $f(x) = \frac{10}{2-x}$ and the x-axis over the interval [–1; 1]. In order to do so, calculate the area of rectangles that approximate the graph from below. How could this method be optimized to calculate the area more accurately?

This example of approximating an area can be solved with the help of the Grundvorstellung "area," as the aim of this task is to calculate the area more accurately, which is possible by showing that the number of rectangles can be increased. Furthermore, it is possible to determine the area more precisely by additionally approx-

imating the graph from above. The issue of approximating the area of a function below the x-axis is not covered by the task.

Potential applications of the definite integral, such as in physics and economics, can be rendered more difficult if the emphasis is on the Grundvorstellung of area, as these applications are considerably more diverse and involve much more than merely calculating areas (Hall 2010, p. 6). "Such an inappropriate 'area-conception' of definite integral stems from the special case, $f(x) \geq 0$, as its construction is based on a generalisation of the special case" (Bezuidenhout and Olivier 2000, p. 78). The area perception is not incorrect, but is in some cases, such as in applying the integral to other problems, less target-oriented or insufficient for understanding the concept of definite integral (Jones 2013, p. 138; Sealey 2006, p. 52) or the concept of integral in multivariable calculus (Dray and Manogue 2006, p. 5). There is a widespread misconception that "integral is area and area is always positive" (Kouropatov and Dreyfus 2013, p. 643). Nevertheless, the Grundvorstellung "area" is often used in textbooks, owing to the advantage of connecting to secondary school students' prior knowledge. However, students may need more time and a deeper understanding, in order to apply the Grundvorstellung appropriately, and to be able to link the aspects "area below a curve" and Riemann sum (Engelke and Sealey 2009, p. 4). Accordingly "area" should perhaps not be used as the first Grundvorstellung for introducing integrals in school lessons.

4.5 The Grundvorstellung "(Re)construction"

By construction or reconstruction in the context of integration, we mean both the (re)construction of a quantity from the given information about rates or speed, and the (re)construction of one of the anti-derivatives of a given function. Existence and variation are important categories for both of these contexts (cf. Hahn and Prediger 2008, p. 178).

The distinction between construction and reconstruction resides in the interpretation of the values of the original function. If one considers them to be the values of a known quantity, then integration amounts to constructing a new functional relationship. If one interprets them as rates of change of an existing functional relationship, then integration amounts to reconstructing this relationship (Bender 1990b).

Example: Spirometer
A so-called spirometer is used by doctors to obtain a curve of the rate of airflow in and out of the lung. Thus, a mouthpiece is used for breathing in and out of the spirometer. This procedure creates a pressure difference in the machine, so that the flow rate of the respired air (in liters per second) can be determined.
The following table shows data for a spirometry measure taken from a resting person during the phase of breathing in (Schmidt 2015).

Time in seconds	0	0.3	0.6	1.0	1.5	1.8	2.1	2.5	2.75	3
Flow rate in liter/second	0	0.18	0.32	0.45	0.50	0.48	0.42	0.28	0.15	0

How much air has been breathed in during the 3-s interval illustrated in the table? What is the average flow rate (in liters per second) during the recorded time of 3 s?

When working on the "Spirometer" example, it is first of all necessary to create a mathematical model for the flow rate, in order to expand the given data to the interval appropriately from 0 to 3 s. We choose a piecemeal linear model as an easy one. For this purpose, adjacent data points are connected by lines. As a result, it is possible to approximately calculate the lung volume that has been inhaled. By taking the maximum flow rate for a short interval and further multiplying it by the length of the interval, we obtain an upper approximation for the inhaled air during that time. Summing the result to the total interval from 0 to 3 s, we then obtain an upper bound for the inhaled air. In this context, the upper sum and the lower sum serve as an illustration, because they are each identified as an upper and lower approximation of the inhaled air volume. The integral is equal to the limit of those upper and lower sums and, hence, represents the air volume in question.

Well-known examples of contexts involving this Grundvorstellung in the sense of quantity reconstruction include the reconstruction of distance travelled from velocity data (e. g. cf. Hußmann 2001, p. 60), or the reconstruction of the net amount of water left over, using data on inflow and outflow for a given container. In reality, in many cases, it is difficult to differentiate between construction and reconstruction. The following claim unites both perspectives: "The function F obtained by accumulation is the same regardless of whether the accumulation is thought of as a new construction from another given function f ... or as the reconstruction of an anti-derivative ..." (Tietze et al. 2000, p. 287).

The Grundvorstellung of a definite integral, as the reconstruction of a quantity, is often accompanied by the summation of partial products, and so is linked to the product sum aspect and the accumulation perception of the definite integral. In the case of area construction or reconstruction, there is a link to both the measure aspect and the area perception of the definite integral concept.

The second perspective of the reconstruction concept contains the idea of (re)construction of an anti-derivative from the data of a given function. The idea of reconstructing anti-derivatives for specific functions can be illustrated geometrically, as an estimate of the slope in small intervals (Herford and Reinhard 1980, p. 98).

The interpretation of the substance of definite integrals from the viewpoint of construction or reconstruction of quantities or anti-derivatives is particularly diverse, and has links to all other aspects of integration. It is therefore regarded as particularly important for understanding the subject matter of integration (Büchter and Henn 2010, p. 92; Danckwerts and Vogel 2006, p. 98 ff.). It is also seen as containing links to interpretations of differentiation: "A fundamental grasp of differentiation

as a local rate of change is crucial for the aspect of integration as reconstruction" (Danckwerts and Vogel 2006, p. 125, see also Sect. 2).

4.6 The Grundvorstellung "Average Value"

The definite integral can also be used to determine average values. The technical basis of the Grundvorstellung of average values is the mean value theorem (cf. Walter 2004, p. 208): if f is continuous over the interval $[a, b]$ then, $\xi \in [a, b]$ such that $f(\xi) = \frac{1}{b-a}\int_a^b f(x)\,dx$. Owing to a conceptual change, this context can be seen as a generalization of the arithmetic mean to continuous functions.

Considered geometrically, the oriented region under the graph of f over the interval $[a, b]$ and the rectangle of width $b - a$ and height $f(\xi)$ have equal area. The value of the definite integral of a given function over a given interval divided by the length of the interval thus gives the "average value" of the function over that interval. This Grundvorstellung is therefore associated with the idea of forming a rectangle with the same area as a given region delimited by a curve.

The mean value theorem cannot, however, be used to define definite integrals. It is therefore a topic of integration that is not an aspect, but rather a substance-based interpretation that gives meaning to the concept of definite integrals. The average value perception of integration can be thought of as the next step in generalizing the arithmetic mean. For example, suppose that we wish to determine the daily average temperature from the data given by an automatic thermometer, which regularly measures the temperature. The mean of the temperature values T_i, $i = 1, 2, 3, \ldots$ at times s_i, $i = 1, 2, 3, \ldots$ can be calculated discretely as follows:

$$T_M = \frac{T_1 \cdot s_1 + T_2 \cdot s_2 + \cdots}{s_1 + s_2 + \cdots}.$$

If we now assume that more data are available, and reduce the time lapse between successive readings, we obtain an expression as a sum for calculating the mean value more precisely. However, there are still fixed amounts of elapsed time between each individual measurement:

$$T_M = \frac{1}{\sum_{i=1,2,3,\ldots} \Delta s_i} \sum_{i=1,2,3,\ldots} T_i \Delta s_i.$$

If every point in time s is assigned a temperature $T(s)$, then we finally obtain the generalized average using integrals:

$$\overline{T} = \frac{1}{s_b - s_a} \int_{s_a}^{s_b} T(s)\ \mathrm{d}s.$$

Emphasizing the average value perception forges a stronger relationship between calculating integrals and stochastic systems, as the expected value of an integrable random variable X, which is defined as $E(X) = \int_\Omega X\,dP$, is the generalization of this type of average (Danckwerts and Vogel 1986; p. 69). Emphasizing this Grundvorstellung too strongly, however, may render other possible applications more dif-

ficult to understand. The average value perception has been judged as of secondary importance for the development of the concept of integral (Tietze et al. 2000; Danckwerts and Vogel 2006). It might therefore be necessary to view the Grundvorstellung in association with the "average rate," as average value thus becomes more important. Bezuidenhout et al. (1998, p. 101) found in a study that students have inadequate intuition about the "average rate" and "average value" concepts. Therefore, we consider this conception to be very helpful in several cases. The second question of the spirometer task serves as an example.

4.7 The Grundvorstellung "Accumulation"

The Grundvorstellung of accumulation builds on a suitable interpretation of product sums that tend toward the definite integral as their limit. In general, accumulation is understood as referring to a process of aggregation, or of collection combined with storage. In the context of integration, the intended meaning is the aggregation or cumulative summation of partial products to form a product sum. An example shows the task "Approximating an Area" (s. Sect. 4.4) in which it is necessary to sum the areas of many rectangles in order to determine the value of the integral.

One example for supporting the accumulation perception is the generalization of the definition of physical work, which can, for the time being, be thought of as the scalar product of force and distance vectors: $W = \vec{F} \cdot \vec{s}$. If the force is now made a function of the distance, in order to calculate the approximate value of this scalar product, the force is assumed to be constant along small segments of the path, and these contributions are summed. This gives the product sum: $W = \vec{F_1} \cdot \vec{s_1} + \vec{F_2} \cdot \vec{s_2} + \vec{F_3} \cdot \vec{s_3} + \ldots + \vec{F_n} \cdot \vec{s_n}$. Taking this idea one step further, one can start with a path-dependent force function, and consider the integral as the generalized product sum: $W = \int_{s_1}^{s_2} \vec{F}\left(\vec{s}\right) \mathrm{d}\vec{s}$.

In the context of the accumulation Grundvorstellung, the definite integral can thus be thought of as a product sum, obtained by accumulating or aggregating multiple partial products. This view of summation emphasizes the process of integration more than the result of the calculation. The geometric perception of the accumulation Grundvorstellung corresponds to the "integral as the limit of (the area of) a series of rectangular regions, as the size of the steps becomes arbitrarily small in the limit" (Blum and Törner 1983). This Grundvorstellung is similar to the area perception, but is more general, as it also evokes other ideas (Blum and Kirsch 1996). From a general perspective, every integral can be considered as an accumulation (Sealey and Oehrtmann 2005, p. 1) and the vector example demonstrates the generality of this Grundvorstellung. Bender (1990a) sees the accumulation Grundvorstellung as providing a relation between the anti-derivative aspect and the Grundvorstellung of area. Firstly, forming sums of products and taking their limit is the converse operation to forming rates of change, and, secondly, finding areas can also be achieved by cumulating subregions. Therefore, the Grundvorstellung "accumulation" occupies an outstanding position (Thompson and Silverman 2007, p. 117). This importance

is thus particularly relevant for the "concepts of the definite and indefinite integrals and its connection with the concept of the derivate" (Kouropatov and Dreyfus 2013, p. 644). The technical essence of the interpretation of integration as accumulation lies in the product sum aspect. It also allows for a link between the process of accumulation and the product to be established.

4.8 Overview: Aspects and Grundvorstellungen of the Concept of Definite Integral

The links between aspects and Grundvorstellungen established from the discussion in the previous sections are shown in Fig. 5, which together describe the concept of definite integral from both technical mathematical and subject-matter didactical perspectives. The left column lists the aspects, and the right column the Grundvorstellungen. The links shown correspond with subject matter relations between aspects and Grundvorstellungen. For example, connecting the concepts of product sum and area means that the latter can be determined with the help of product sums according to the Grundvorstellung "area." The missing connection between "measure" and "average value" should state that from a subject-specific point of view, the measure aspect has no meaning for the Grundvorstellung "average value."

The characterization of aspects and Grundvorstellungen for integration are not used consistently across the literature. In particular, there is often no differentiation between aspects and Grundvorstellungen. Hußmann (2001) distinguishes, for example, between *aspects* that permit "different angles for viewing the concept of integral" and Grundvorstellungen, among which he includes both *accumulation* and the *total effect,* as do Danckwerts and Vogel (2006, p. 96 ff.), calling them "underlying conceptions". Tietze et al. (2000) uses the term *Grundverständnis* (basic understanding), and states, regarding the calculation of definite integrals: "The integral $\int_a^b f$ of a function f over an interval $[a, b]$ is a number resulting from a limiting process, giving the result of an accumulation." (Tietze et al. 2000, p. 281). The Grundvorstellung of accumulation is thus accorded special importance for definite integral calculations. Accumulation is also considered to have many links to the

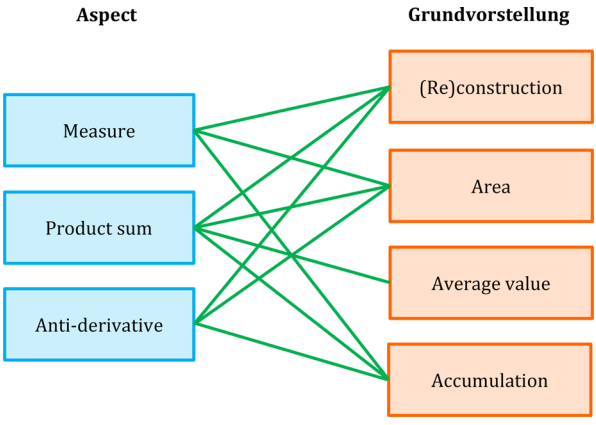


Fig. 5 Aspects and Grundvorstellungen of integration

various different aspects: "Anti-derivatives arise from accumulation," and "Finding the area under the graph is accumulation." (Bender 1990b, p. 114 ff.). It is seen as a concept or an idea (Thompson and Silverman 2007, p. 117; Kouropatov and Dreyfus 2013, p. 643). Büchter and Henn (2010, p. 92) on the other hand, focus on a *fundamental idea*, and regard *reconstruction* as the fundamental idea behind integration.

In addition to the aspects and conceptions mentioned earlier, Hußmann (2001) also described the approximation aspect, both "with regard to approximating areas and with regard to approximating the function" (Hußmann 2001, p. 56). Approximation is also seen in an example of a well-formulated curriculum as the "initial activity for introducing the concept of the definite integral" (Kouropatov and Dreyfus 2013, p. 646).

In the theory described here, approximation is neither an aspect nor a Grundvorstellung of integration. Approximation on its own cannot be used for defining definite integrals. The definition occurs instead within the framework of the product-sum aspect described earlier, as the limit of upper and lower sums. Thus, product sums have been described as the relevant aspect, not approximation. The idea of approximation can occur in all of the previously mentioned Grundvorstellungen. Therefore, it is not useful to specify "approximation" as a Grundvorstellung on its own.

The idea of approximation is important for developing a conceptual understanding of the definite integral (Sealey and Oehrtman 2005, p. 83). Integrating, as a means of determining areas under curvilinear boundaries, also cannot be considered without the idea of approximation (Dankwerts and Vogel 2006, p. 100 ff.).

In any case, it is important in the mathematics classroom that there be varied views on the definite integral, "to build a more insightful concept" (Rasslan and Tall 2002, p. 8). Huang (2012, p. 167) regards narrowing down viewpoints to one or more specific aspects or Grundvorstellungen of integration as questionable.

5 Aspects and Grundvorstellungen in the Context of Concept Understanding and Subject Matter Didactics

In this section, the *Aspects* and *Grundvorstellungen* of differentiation and integration, established in the previous sections, are classified according to the process of concept development. We propose two models of understanding for the concepts of derivative and integral, based on our subject matter related analysis of aspects and Grundvorstellungen.

To describe the level of proficiency in more detail, we add four categories of concept understanding based on Vollrath (1984). This third dimension – with aspects and Grundvorstellungen as the first and second dimensions – helps link the ideas of concept image and concept definition (cf. Fig. 1).

The level of *intuitive concept understanding* enables us to use simple examples and to compare different representations. The level of *subject matter concept understanding* allows for a concentration on properties and characterizations – in different representations – of the concept. The level of *integrated concept understanding* en-

ables seeing relations to other concepts, as well as relations between properties of the concept. The level of *critical concept understanding* enables the indication of formal definitions, argumentations, and proofs.

5.1 The Relation Between Aspects, Grundvorstellungen, and Concept Understanding for the Concept of Derivative

Fig. 6 shows the relation between aspects and Grundvorstellungen in differentiation. Overall, this can be represented by a 2 × 4 matrix (with six non-empty cells). If the "dimension" of concept understanding is added for each of the four categories, this can be represented by a 3D model.

Using this representation, 6 × 4 of the "cells" can be characterized. In the next section, some of these cells will be described as examples at the level of intuitive concept understanding, because the different pairings of Grundvorstellungen and aspects can be clearly identified.

5.1.1 On the Concept of Derivative

- At the level of *intuitive concept understanding,* the characterization of the cells is given as follows:
  - The Grundvorstellung of local rate of change with the limit aspect encompasses the conception that the average rate of change stabilizes numerically, as the interval declines in size.
  - The same intuitive limit, together with the interpretation of differential quotients as the slope of secant lines, yields the Grundvorstellung of tangent slope within the same aspect.
  - Given the aspect of local linearization, the conception of tangent slope is that of closely fitting lines to the graph.

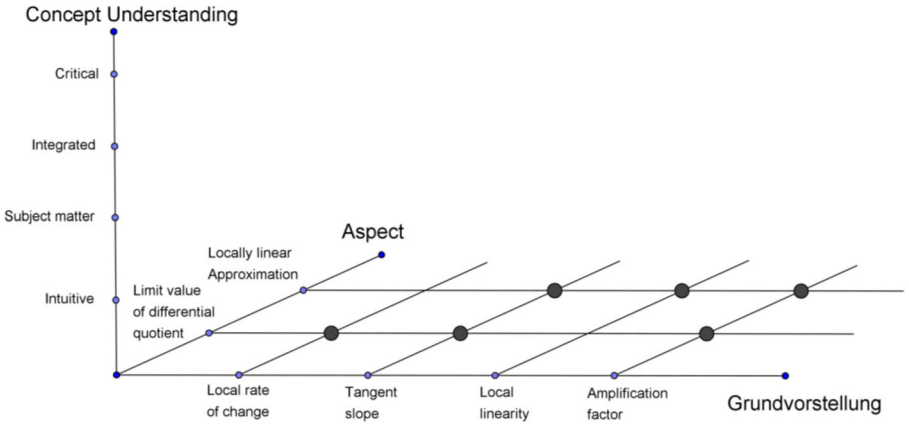


Fig. 6 3D matrix representing aspects, Grundvorstellungen, and concepts understanding for the concept of derivative

- The Grundvorstellung of local linearization using a function microscope (cf. Kirsch 1979, p. 25) can be viewed as an intuitive understanding of the idea of local linear approximation.
- The intuitive conception of amplification factors incorporates, among other things, experiences beyond the field of mathematics – small changes in independent parameters are transferred to the dependent parameter. The conception of a generalized factor of proportionality can be identified within the differential quotient aspect. However, within the aspect of linear approximability, the calculation-based perception of a factor as in $\Delta y \approx m \cdot \Delta x$ is dominant.

At the levels of understanding further along the vertical direction, the various different Grundvorstellungen are integrated and combined. Concept understanding is enhanced and extended so that Grundvorstellungen help reflect important mathematical facts.

- *Subject matter concept understanding* involves in particular the application of the concept and its properties relating to reasoning processes. For example, using the aspect of limits of differential quotients, it is possible to reason (based on a calculation) that tangent slopes combine additively. Using the concept of rate of change within the aspect of local linearization, it is possible to explain the product structure of the chain rule (see Sect. 2.6).
- Results about monotonicity and local extrema can be interpreted at different levels of representation. However, these different conceptions link together increasingly. Graphical interpretation, for example, links the rates of change and the conception of tangents. Thus, we arrive at an *integrated concept understanding*. If an understanding of this type takes root, it becomes possible to embed the concept of derivative into the wider theory of calculus as a tool for studying functions.

The formation of a *critical concept understanding* also makes it possible to argue that certain functions are not differentiable, for example, because they do not appear to be locally linear, either because there is no unique factor of amplification, or because a limit value does not exist. We take the well-known function $f : \mathbb{R} \to \mathbb{R}, x \longmapsto \sqrt[3]{x}$ as an example. The fact that f is not differentiable in $x_0 = 0$ can be understood with the help of different Grundvorstellungen. We can recognize that a vertical straight line has no gradient (tangent slope would be infinite), we cannot find any linear function that approximates locally at best (local linearity, all finite slopes are too small), and we find that small changes of x near 0 cause discontinuous changes (amplification factor would be infinite). We recognize that each Grundvorstellung has a specific way of explaining non-differentiability in this example. This does not contradict, however, the tendency that different Grundvorstellungen are integrated better at higher levels of understanding.

This 3D model can be used in several ways. On the one hand, it could support teachers in introducing this important topic of derivative. Thus, one has an overview of the Grundvorstellungen and aspects at the addressed level of concept understanding. On the other hand, this model can also be used for diagnostic intervention. Thus, one also has an orientation, while constructing examples for a certain test on – here – derivatives. But this model can guide educators of educators, in order to

show preservice teachers, at a certain point, what needs to be known, so that mathematics education moves forward in its orientation. All the listed facts can also be interpreted for the integral. We therefore consider the aspects, Grundvorstellungen, and concept understanding analogously.

5.2 The Relation Between Aspects, Grundvorstellungen, and Concept Understanding for the Concept of Integral

Fig. 7 gives the relation between aspects and Grundvorstellungen for integration. In the plane, this relation is given by a 3×4 matrix with 10 non-empty cells. If the "dimension" of concept understanding is added for each of the four categories, this can be represented by a 3D model.

- The relations between Grundvorstellungen and aspects within an *intuitive concept understanding* can be described with the following example:
  - The Grundvorstellung of area, combined with the measure aspect, includes the conception that integrals can be used to measure the area of regions between the graph of the function and the x-axis.
  - The Grundvorstellung of reconstruction, combined with the antiderivative aspect, includes the idea that an antiderivative can be constructed graphically from a given function, using piecewise-linear curves.
- At the level of *subject matter concept understanding*, one can, for example, use the product sum aspect to argue within the context of the average value Grundvorstellung that the integral can be used as the basis for a generalized concept of average value.
- The fundamental theorem of calculus should be seen as part of an *integrated concept understanding*. On the one hand, this can be understood from the angle of the antiderivative aspect, using the accumulation Grundvorstellung. Alternatively, working from the perspective of area together with the measure aspect also makes sense.

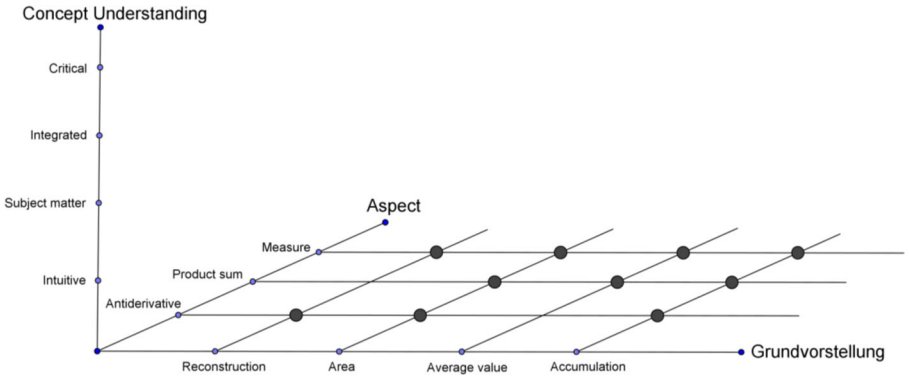


Fig. 7 3D matrix of understanding the aspects, Grundvorstellungen, and concepts within the concept of integral

- Forming a *critical concept understanding* makes it possible to reason, using the antiderivative aspect together with the Grundvorstellung of construction that certain functions do not possess an antiderivative.

6 Final Remarks

The aim of this article was to specify the concept of Grundvorstellung for differentiation and integration, and embed it in a model of understanding for these two concepts. This was accomplished by a technical and subject-matter didactic analysis that established the (technical) aspects of these two concepts. The result is a representation in the form of a three-dimensional matrix, whose cells are described by a triple (aspect, Grundvorstellung, level of concept understanding). Each cell can be characterized on the one hand by specific knowledge, abilities, and the capacity to manipulate the concepts in question, or alternatively, by the nature of the relation between the technical properties (aspects) and Grundvorstellungen. This leads to a description of the expected competencies for each of the "cells" of the 3D matrix. This 3D matrix can also be interpreted as the basis for a competency model that undertakes to establish a "theoretical delimitation of subject matter and structural description of an area of competency" (Leuders 2014, p. 9), in this case of the concepts of derivative and integral. Furthermore, this can be a first step toward an empirical evaluation of models of understanding on the basis of aspects and Grundvorstellungen.

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