REASONING WITH DYNAMICALLY LINKED MULTIPLE REPRESENTATIONS OF FUNCTIONS

Guido Pinkernell

University of Education Heidelberg, Germany

In video-taped interviews teacher-students were asked to describe and explain effects of a parameter a *on the standard representations of* $f(x)=x^2+a$ *in a computer based dynamic learning environment. An analysis of one particular interesting yet typical misconception leads to differentiating between surface perception and structural insight. Theoretical considerations based on Duval and Davydov lead to postulating that for a full understanding of the relation between* a *and* $f(x)=x^2+a$ *a learner needs to identify to structural analogies between the representations of* f*. A qualitative analysis of further interviews results in a category model of student responses that can be used for diagnostic purposes.*

Keywords: multiple representations, abstraction, ICT, qualitative analysis

INTRODUCTION

A typical task connected with the use of mathematical software in classroom is to explore the relations between the standard representations of functions. For example, pupils could be asked to explore the effects of a parameter a in $f(x) = x^2+a$ on the shape and position of the graph of f by means of a dynamically linked multiple representation learning environment, where the value of a is controlled by a slider (fig. 1).

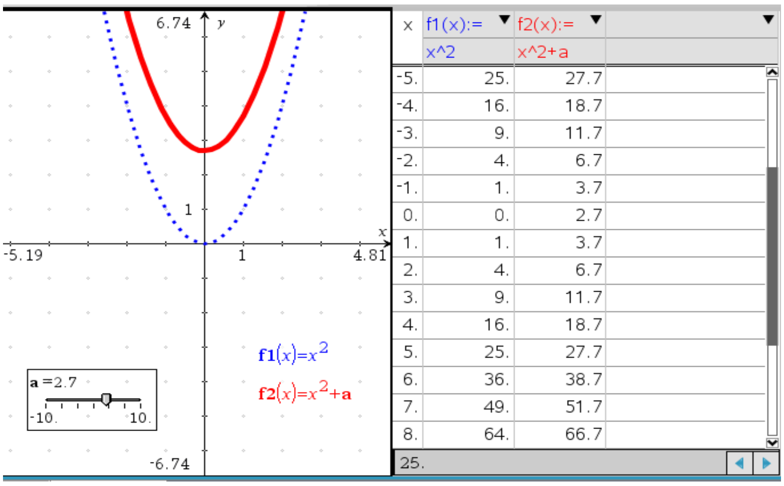

Figure 1: Dynamic multiple representation environment for exploring the effects of a on the representations of $f(x)=x^2+a$.

Usually, it seems sufficient to observe that changing the value of a causes the parabola to move upwards while a gives the distance. But is it really as simple as that? Just describe what seems, literally, obvious?

Duval (2002) argues that for showing a full understanding of the concept of function, a learner needs to be able to change within and between various representations of a function, e. g. equation, table and graph. This means that properties of one representation are explained by properties of another. If a learner is not able to perform such a change then, following Duval's rationale, he does not understand to the full extent, even if his observations within one representation appear to be perfectly valid.

This article begins with a case study that illustrates Duval's concept of understanding functions. A student describes the effects of the parameter a on the graph of $f(x) = x^2 + a$ (fig. 1). She then realises that one of her observations contradicts what she has learned about how graphs and equations connect, but she is not able to resolve this contradiction. She knows a lot, yet she does not quite understand.

So it appears that learning about functions based on visual perception only is not sufficient. For developing a sound understanding a learner needs to turn his attention from the visual properties of the different representations to structural analogies between them. These analogies do not equate to perceivable similarities between the representations. For identifying structural analogies one needs to “see through” their specific appearance in each representational mode. In this sense, the identification of structural analogies is a form a abstraction. Hence, after the case study, this article introduces the concept of learning by scientific abstraction by Davydov (1972). It serves as a suitable theoretical basis for justifying our postulation that learners need to refer to a structural level.

When theoretical considerations lead us to expect that learners behave in a specific way then we must be certain that they are able to do so. The last part of this article reports on a qualitative analysis of further interviews with teacher-students where three categories of understanding could be identified, among them references to structural analogies between representations as required by theory.

A CASE STUDY

In non-standardised videotaped interviews, teacher-students of the University of Education Heidelberg were given the task as shown in fig. 1. They first were asked to describe the changes within graph, table and term when the slider is operated. Then they were asked to explain why they thought their descriptions were correct. One student described the effects of the slider as both a translation and a dilation of the parabola:

1	Student:	[moves slider to the right] well the parabola moves upwards along the y-axis … the um how is it called the width [moves both palms repeatedly towards each other as if clapping hands]
2	Interviewer:	yesyes, ok
3	Student:	changes … anyway when one moves it to the right … towards the positive … [moves slider to the far right such that the parabola nearly vanishes from the screen] … and when one moves it downwards to the negative [moves slider to the left such that the parabola's vertex nearly touches the lower screen edge] and here the parabola becomes wider but still opens upwards
4	Interviewer:	can you explain why the parabola moves up or down when one changes the slider value [points at the slider with a pen]

5 Student: [looks a the slider, murmurs] hm what is a … what … is a [leans back, talks louder] a is … a was something … a is … the y-direction

6 Interviewer: ah

7 Student: when I … upwards … well then it is not a normal parabola any more then it is, like, somehow narrower

When asking the student to explain her observations (line 4) the interviewer, who is the author of this text, ignores that she describes the effects of a as a translation as well as a dilation of the curve (lines 1 and 3). The student answers by referring to knowledge about the effects of a she probably had acquired at school (line 5). But, again, she immediately points out that the curve looks shrunk (line 7). This seems to be a problem for her as it conflicts with what she has learned about parameters and graphs (meanwhile the value of a has been set to 1):

8 Student: this is simply a normal parabola that has been moved upwards by one, but the width changes too.

9 Interviewer: why do you say 'but'?

10 Student: the parabola does not move upward only but it becomes narrower too, and then there should be something in front of the [x] squared

11 Interviewer: ah, why?

12 Student: because that's how I learned it

That the parabola has been reduced in size is, with respect to the algebraic structure of x^2+a, wrong. The student is aware hereof since she expects a factor in front of x^2 (line 10) to be able to speak of a dilation. Yet, considering the graph only, she proves to be a careful observer. The parabola does appear to be shrunken (fig. 2):

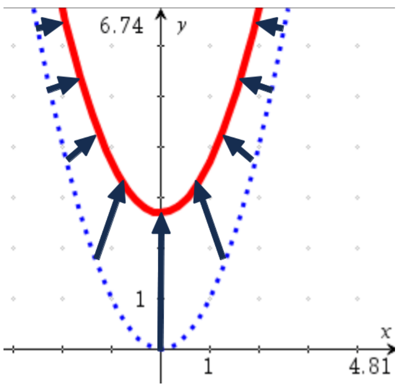

Figure 2: Moved upwards only or shrunken too?

The student realises that her description contradicts what she knows about the effects of a parameter change on the function graph. To resolve this contradiction she needs to explain the effects of a on the graph by means of the specific algebraic properties of the equation, i. e. she needs to perform a coherent representation change. It is the additive structure of the expression x^2+a that dismisses the interpretation of a dilation and supports the translation: Every single value of x^2 is increased by a, so each single point $P(x|x^2+a)$ of the parabola is moved upwards by a. So a valid interpretation of the effects of a parameter change on the representations of a function needs to be based on an analysis of the term structure. Hence algebra plays a decisive role for describing and explaining relations between parameters and functional representations. When a learner describes the effects of a parameter change without giving a sufficient explanation on the base of the algebraic representation it is not clear whether his descriptions are based on his visual impressions only or whether they are based on a structural insight into the situation. This is particularly problematic when these descriptions fulfil the expectations of the teacher: If a learner only mentions a vertical translation by a, does he just reproduce what he sees, or does his description reflect an understanding of the `mechanism´ of how a affects each representation of x^2+a? One risks confusing a correct verbal description with an understanding of the situation, pupils as well as teachers.

UNDERSTANDING MULTIPLE REPRESENTATIONS BY ABSTRACTION

Does the student from the case study understand the effects of a on the representations of x^2+a? She gives a careful description of what she sees on the screen. And she knows, too, how the effects of a parameter on a function graph can be read from a given equation. But she cannot resolve the conflict between her observations and her knowledge. In this sense, she shows a lack of understanding.

So what does she – or does a learner in general – need to achieve so that he or she shows a full understanding of the relation between parameters and the representations of function? Giving a description of his observations only is not sufficient as we have argued, even if the description is correct. He needs to give reasons for why his observations are valid, i. e. why they are consistent with the whole of the multiple representation environment. Reasoning by referring to well known rules might be acceptable, but in the case of the student more was needed to clarify the conflict between what she saw and what she knew. It would be helpful to explain the mechanism of how a and representation of x^2+a connect. This means to refrain from reproducing visual information but to analyse analogies between the representations on a structural level. Focussing on structures instead of surface leads to a cognitive activity that is central for the learning of mathematical concepts: abstracting.

In the context of learning with a dynamic multiple representation environment abstracting means extracting the essential information from representations by conceiving structural analogies between representational forms while eliminating irrelevant surface properties. By referring to the concept of scientific abstraction by Davydov (1972) and others we will show that conceiving structural analogies is achieved by identifying invariants in the dynamic multiple representation environment while the structure of the algebraic representation is decisive.

Mathematical concept formation as the result of abstraction

Mitchelmore and White (2007) identify two different approaches to abstraction among theories of mathematical learning, which they call empirical and theoretical.

<u>Empirical abstraction</u> refers to a cognitive representation of knowledge that results from identifying common properties in a set of examples. “Abstracting is an activity

by which we become aware of similarities [...] among our experiences. [...] An abstraction is some kind of lasting change, the result of abstracting, which enables us to recognise new experiences as having the similarities of an already formed class" (Skemp 1986). However, empirical abstraction that is limited to a perceptual analysis of real or cognitive objects can hardly explain the formation of such concepts that meet the scientific requirements of generality and precision. For Stern & Schuhmacher (2004), empirical abstraction is typical for childlike learning, where formation of a new concept is based on perceptual activities. A whale appears to be a fish because it has fins and lives in water. That whales are biologically closer to man than fish is incomprehensible when one follows his visual impression only. Hence, for the formation of a scientific concept, a theoretical basis is needed which supports argumentation that is independent from perceptual evidence.

Theoretical abstraction: To be valid beyond experience, knowledge needs to be developed within a theoretical system of its own, which comes with specific symbolic representations and rules of argumentation. Following Vygotsky (1934/1986), this symbolic representation form does not need to resemble any physical features of the knowledge that it represents. In fact, perceptually or otherwise empirically accessible properties are unsuitable for forming an abstract concept. "A theoretical idea or concept should bring together things that are dissimilar, different, multifaceted, and not coincident, and should indicate their proportion in the whole [...] Such a concept, in contrast to an empirical one, does not find something identical in every particular object in a class, but traces the interconnection of particular objects within the whole, within the system in its formation." (Davydov 1972/1990, 255)

For continuing our case analysis the concept of theoretical abstraction – or "scientific abstraction" as Davydov puts it (1972/1990) – appears to be suitable: The effects of a on x^2+a can be described as a function $a \rightarrow x^2+a$, which is a function different from f. Its properties can only be derived from changes within the representations of f. The change within the graphical representation of f appears to be a translation and a dilation, here the student is perfectly right. To decide whether this interpretation conforms with the rest of the multiple representation environment, the algebraic expression of $a \rightarrow x^2+a$ needs to be analysed. Its additive structure decides which of the two interpretations of the effect of a on x^2+a is valid. So it is knowledge about symbolic algebra that forms the necessary theoretical basis for understanding relations between the multiple representations of a dynamic learning environment. However, the student is not able to apply her knowledge about algebra to explaining how, or whether at all, her descriptions are valid for the whole of changes within the multiple representations of $f(x) = x^2+a$.

Abstracting as conceiving structural analogies

In a teaching concept of "ascending from the abstract to the concrete" based on Davydov, Giest (2011) states that, with each new learning process, "initial abstractions" are gained from examining a learning material that allows change and variation. From varying the material, invariants become apparent that initiate the necessary reasoning for identifying a constant structure within change. With a dynamic multiple representation environment (e. g. fig. 1) the necessary variation here is twofold. First by operating the slider, thus changing the visible appearance of each representation and second, by switching between the three representation forms. These changes correspond to Duval's (2002) forms of representational changes that characterise a full understanding of the concept of function.

Obviously, the student from the interview meets the first of Duval's requirement at least partially. Within the graphical representation, she gives two pertinent interpretations of the effects of the parameter change on the actual function graph, and from the algebraic representation she can read correct information about the effects of parameters on function graphs in gene. But she does not fulfil the second criterion of Duval. She is not able to make a coherent change between the algebraic and the geometric form of representation here. Or to say it with Giest: She is not able to identify the necessary invariants within the multiple representation environment.

Conceiving structural analogies as identifying invariants

In fig. 1, the invariant in question is not visible, it becomes apparent as changes between and within representational forms. Considering the additive structure of the term of x^2+a, the invariant is characterised by the common operator $+a$. In the equation, the invariant is the summand $+a$ that redefines $f1(x)=x^2$ to $f2(x)=x^2+a$ (fig. 3). In the table, the invariant is the constant difference between the values of $f1$ and $f2$ in all table lines. In the graphical representation, the invariant is the constant vertical distance between the two graphs of $f1$ and $f2$, which is the same at all points. Thus, the invariant has a specific meaning in each representation form, yet, in each form, it can be visualised by an arrow with constant direction and length. Especially the arrows from the geometric representation form show that the effect of a on the graph of x^2+a must be interpreted as a translation and not a dilatation.

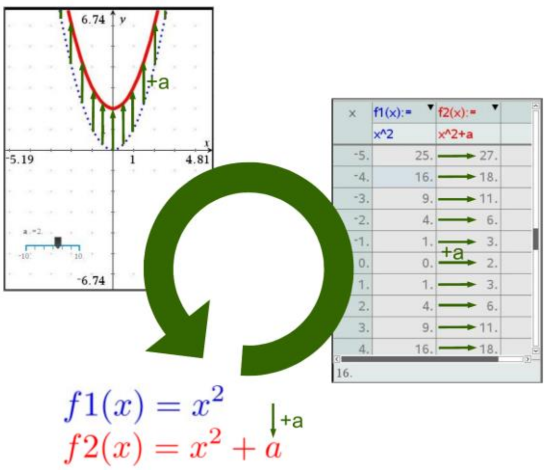

Figure 3: The invariant $+a$, identified in all three representation forms, shows that the effect of a on x^2+a is indeed a translation.

We can sum up now: To show understanding of the relation between a parameter a and the representations of f(x)+a a learner needs to identify the operator +a as an invariant within each representation and between all representation forms of f.

This seems expecting much from pupils. To facilitate identifying structural analogies one could, of course, consider adding visual aids like the green arrows in fig. 3. However, there are various forms in which learners can indicate that they have indeed "seen through" the surface of the representations. Next we report on a study where references to structural analogies in further interviews were identified.

CATEGORIES OF REASONING: AN INTERVIEW STUDY

Aim and methods

Together with the interview from our case study further interviews were analysed with the aim of categorising students' answers regarding to what extent they showed a structural insight into the relations between graph, table and term of f. The interviewees were teacher-students of the University of Education Heidelberg from their first to their third year of study, all having selected mathematics as one of their compulsory subjects. The interviews contained questions about various tasks about for exploring the relation between parameters and quadratic functions. The task from fig. 1 was the first. All were accompanied by dynamic multiple representation learning environments, prepared in advance by means of the TI-Inspire CAS software on a laptop. Apart from two question – the initial one that asked for a description of how the given parameter affected the appearance of graph, table and equation and one that asked the students to explain why they thought their observations were correct – no other questions were fixed in advance. Thus non-standardised, the interview left enough room for following the line of thoughts freely.

For the analysis, five interviews were selected which, on first view, promised a sufficiently large range of students' observations and explanations. The categories were developed by means of a qualitative content analysis (QIA) in the form of a deductive category application (cf. Mayring 2000 and 2010). The QIA is a systematic method for text analysis guided by pre-set coding rules. In the variant of the deductive category application these coding rules are derived from the relevant theoretical framework. These rules are then applied repeatedly to the text by trained coders with the aim of enhancing inter-subjective comprehension of the rules. In our case these were three teacher-students who did not take part in the five interviews selected for coding. They were able to understand the math specific terminology of coding rules while, as students, they relied on very precise formulation of the rules to agree on common coding results.

Results and discussion

Apart from the categories "reason with reference to a rule" and "reason with structural reference" which both derive from the theoretical considerations above, a third category emerged during the coding process. Also, the well-known Toulmin model of argumentation was introduced between coding rounds for a better differentiation between a student's description of the effects of the parameter a ("claim" in the Toulmin model) and an explanation of these observations that made references to a rule ("warrant" in the Toulmin model). The third and most successful coding round in terms of intercoder reliability was based on the following rules:

- Category A "Proposition": All statements or gestures that can be seen as answers to the interviewer's initial prompt "Describe what you see when you operate the slider" and that show characteristics of a general rule. They refer to the effects of changes of the slider on the representations of the given function. If the interviewee modifies his propositions later during interview, these modifications are coded within this category too.
- Category B.1 "Reason with reference to a rule": All statements or gestures that can be considered as reasons for the observations assigned to category A or their modifications and that refer to a rule or sound like one. Such rules often refer to connections between the parameter and the curve's shape or position. The statement does not refer to single parameter values but shows that the interviewee implies a global validity. Such rules do not need to be stated explicitly but the interviewee can also indicate that he knows of such a rule (e.g. "that's how I have learned it"). Even when the interviewee shows understanding at a structural level in some other part of the interview, reasons with reference to rules or knowledge are to be assigned to this category.
- Category B.2 "Reasons with reference to an example": All statements or gestures that can be considered as reasons for the observations assigned to category A or their modifications and that refer to the actual value of the parameter or the actual state of the visible graphical configuration. These statements often refer to connections between the actual value of the parameter and the curve's present shape or position. Even when the interviewee refers to rules or shows understanding at a structural level elsewhere, reasons with reference to rules or knowledge are to be assigned to this category.
- Category B.3 "Reasons with reference to structure": All statements or gestures that can be considered as reasons for the observations assigned to category A or their modifications and that apparently refer to an invariant between different representation forms or within one representation. The invariant here is the value of a which takes on a specific appearance in each representation: In the algebraic representation it is the summand $+a$, in the numerical representation it is the difference between the old and new function value in each table line. In the geometric representation it is the vertical distance between the graphs of both functions at each point. It differs from category B.2 insofar as statements here do not refer to single values of a but to any value of a in general. It differs from category B.1 insofar as a has been globally (i. e. for all values of x) identified within a representation.

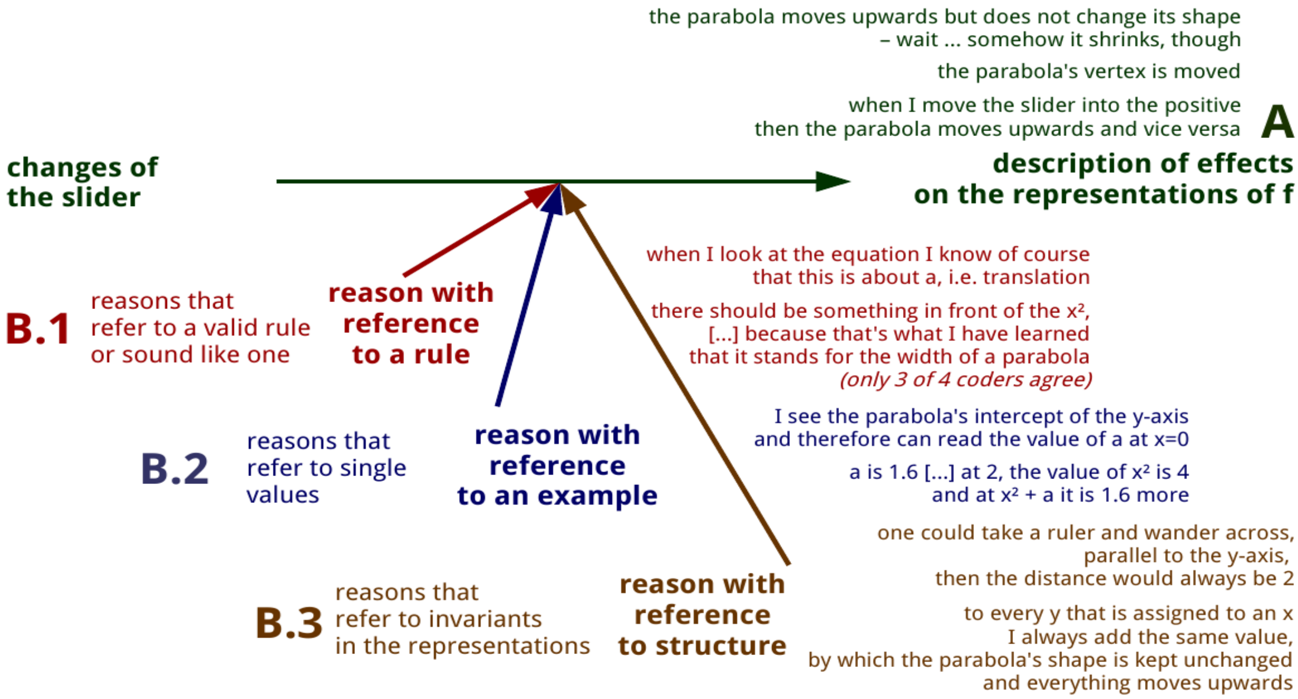

Figure 4: Categories of reasoning while exploring the effects of a parameter a on the representations of x^2+a in a computer based learning environment

The layout of fig. 4 places the four categories at appropriate places within the Toulmin model of argumentation. The proposition (category A) corresponds to Toulmin's "claim" which are descriptions of the effects of slider changes. The three categories of reasoning (B.1, B.2 and B.3) are placed as warrants into the diagram. All categories are illustrated by statements taken from interviews after little linguistic polishing. Generally a statement was considered exemplary when all four coders including the author agreed. One statement for category B.1 is an exception which, to the author, still appears to be a significant example for this category.

In our opinion the three categories B.1, B.2 and B.3 cover a reasonable bandwidth of pertinent explanations of the effects of a on the representations of x^2+a. Here, category B.3 "reason with reference to structure" represent an understanding of the relations between representations of a function best. One student mentions a ruler which, when moved vertically across the coordinate plane, would measure a constant distance between the graphs of $f1$ and $f2$. Another student pointed out that, for all x, a is added to the corresponding $f(x)$, by which he explained the congruence preserving translation he had observed. Category B.2 "reason with reference to an example" represents statements that explain the effect of a for a single value of x, mostly this is $x = 0$ or the curve's intercept of the y-axis. These do not belong to B.3 since they do not explain the congruence preserving translation of the graph. Category B.1 "reason with reference to a rule" contains statements that refer to knowledge about relations between parameters and representations of functions that the students previously have acquired or that they considered of to be generally true.

The category system from fig. 4 covers responses from the case a → x^2+a only, where the invariant is easily identified as the constant vertical distance between the two curves or the constant difference between the two function values, each illustrated by an arrow with constant length (fig. 3). With parameters in other places of the algebraic expression this is different: For example, with b → b·x^2 or c → $(x+c)^2$ other (mis)interpretations of the effects of the parameter can be expected. In recent interviews students refer to a "vertically moving column" when they observe the effects of c on the table of $(x+c)^2$. Or, with d → x+d, students describe the effects of d on the linear graph as "moving diagonally from bottom right to top left".

Prospects and consequences

Presently, standardised interviews are being developed for diagnostic purposes based on these results. Apart from diagnostic use, these results may be significant for classroom teaching too. Nearly all students and, in recent interviews, pupils reported that they had little, if any, experience with computers in school. Many were not able to explain the movements on screen which, to them, were totally new. These results plead for a more extended use of dynamic software in mathematics teaching. In a dynamic environment, insufficient conceptions about function representations become apparent and can be dealt with openly. Last, these results show that, for an exploratory learning with computer based dynamic multiple representations too, a sound basic knowledge in algebra is necessary. Knowledge about term structure turns out to be essential as it plays a decisive role when validating the explorations.

REFERENCES

Giest, H. (2011). Wissensaneignung, Conceptual Change und die Lehrstrategie des Aufsteigens vom Abstrakten zum Konkreten. http://www.ich-sciences.de/index.php?id=135&L=0%2529 (last access 13/02/27).

Davydov, V. V. (1972/1990). Types of generalisation in instruction. Reston: National Council of Teachers of Mathematics. (Originally published in 1972).

Duval, R. (2002). Representation, Vision and Visualization: Cognitive Functions in Mathematical Thinking. Basic Issues for Learning. In F. Hitt (Hrsg.), *Representations and Mathematics Visualization* (S. 311–353).

Mayring, P. (2000). Qualitative Content Analysis. http://www.qualitative-research.net/index.php/fqs/article/viewArticle/1089/2385 (last access 14/09/13).

Mayring, P. (2010). Qualitative Inhaltsanalyse (11th ed.). Weinheim, Basel: Beltz.

Mitchelmore, M., & White, P. (2007). Abstraction in Mathematics Learning. *Mathematics Education Research Journal*, *19*(2), 1–9.

Stern, E., & Schuhmacher, R. (2004). Lernziel: Intelligentes Wissen. *Unversitas*, *59*(2), 121–134.

Vygotsky, L. S. (1934/1986). Denken und Sprechen. Frankfurt am Main: Fischer-Taschenbuch-Verlag.