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Using patterns generically to see structure

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Using patterns generically to see structure

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A common approach to generalization activities in school algebra is to set them in the context of figurative patterns (e.g., a row of touching squares formed of matchsticks, with a relationship between the numbers of squares and matchsticks; or a row of touching tables surrounded by chairs, with a relationship between the numbers of tables and chairs). This approach allows students to see and explain numerical relationships in terms of the figurative pattern's structure. However, patterns are all too often presented, in textbooks and by teachers and researchers, in the form of a systematic sequence of elements, with the result that students generate a systematic set of ordered pairs for which they try to induce an empirical relationship, divorced from the structure of the pattern that produced them. In this paper, I argue that one reason that students appear to have difficulty in seeing structure is that they are not sufficiently initiated into this way of thinking. Further, I argue that the search for structure can often be done by focusing on an individual element of a pattern and treating it as a prototype, that is, generically. I begin by looking at some of the influences that might have fostered an empirical approach and point to manifestations of this approach in a recent special issue of *ZDM – The International Journal on Mathematics Education*. I then discuss how well various generalization tasks are suited to a generic approach. Last, I report on a short intervention where students were encouraged to use a generic approach, with positive results.

Keywords: algebra; generalize; structure

A number-pattern-spotting approach in English schools: some possible reasons

The English National Curriculum

It has often been stated that generalizing should be at the heart of mathematical activity in school (e.g., Mason, Johnston-Wilder, & Graham, 2005). It is thus rather surprising that the act of generalizing plays such a small part in the algebra component of the English National Curriculum. Students are required to work with 'generalized numbers' (in some sense or another; see Küchemann, 1978), but rarely are they asked to *make* generalizations. Where they are, this is commonly in the context of sequences, with the aim of finding a general relationship between a term in a sequence and its position in the sequence. Thus, for example, it is recommended that in Year 8 (Grade 7, age 12–13), students 'use linear expressions to describe the *n*th term of a simple arithmetic sequence, justifying its form by referring to the activity or practical context from which it was generated' (Department for Children, Schools and Families [DCSF], 2008, p. 8). Leaving aside the peculiar notion that students should only encounter simple sequences and only ones that are arithmetic (i.e., linear), this statement

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contains the laudable aim of identifying the form of a sequence (i.e., its structure) through the context from which it arose. However, tasks of this type are commonly tackled by systematically generating the first few terms of the sequence and putting the values in a table. This distances students from the structure of the activity that generated the sequence and often leads to students merely seeking an empirical rule for the numbers they have found, that is, to making an empirical rather than a structural generalization (Bills & Rowland, 1999). Moreover, this rule is quite likely to be recursive (i.e., term-to-term) rather than 'functional' or closed (i.e., position-to-term), thereby providing a restricted description of the general relationship.

Studies can be found that have deliberately adopted a generic approach but these are relatively rare. One such is by Swafford and Langrall (2000), which suggests that when successive elements of a pattern are presented, this is more likely to result in a term-to-term generalization than when pattern elements are not ordered. Often, researchers have promoted (unwittingly, perhaps) an incremental, step-by-step approach. Thus for example, Orton and Orton (1994), who had anticipated that asking students to build actual shapes out of matchsticks would help them see structure, report rather wistfully:

> It had been hoped that the experience of actually handling the matches and building *the next shape* [emphasis added] would help pupils to focus on the matches and make use of the structure of the shapes but, once the numbers had been made explicit, it often appeared that the matches were set aside. (p. 413)

MacGregor and Stacey (1992), writing some time ago about the situation in Australia, state: 'One common textbook approach takes a geometric design, immediately derives from it a table of values and then seeks an algebraic formula which will produce the numbers in the table' (p. 369). It is interesting to consider how this way of tackling generalization tasks has come about (and why it is still with us). In England, it may in part be because the current curriculum emphasizes *sequences*, rather than general relationships per se. This is particularly the case in the highly influential *Framework for teaching mathematics* document (Department for Education and Employment [DfEE], 2001), which provides guidance to teachers (and textbook authors) by exemplifying the condensed statements that define the statutory content of the curriculum. In fact, the National Curriculum hardly mentions relationships at all, though it does refer to functions and formulae. These, however, are artificially kept apart, with the algebra curriculum split into 'equations, formulae and identities' and 'sequences and functions'. Figurative patterns could provide a challenging context for work on formulae but they rarely appear in this area in the *Framework* document, though Figure 1 shows one, rather unengaging, example (for the learning outcome 'substitute positive integers into simple linear expressions').

The expression $3s + 1$ gives the number of matches needed to make a row of s squares.

How many matches are needed to make a row of 13 squares?

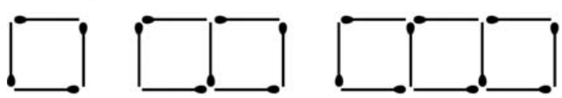
Figure 1. Exemplification of a Year 7 learning outcome (DfEE, 2001, p. 138).

As in this example, most of the emphasis in the National Curriculum is on using formulae rather than on constructing them. The work on sequences seems to start in a similar way with the emphasis primarily on using rules (term-to-term and position-to-term) to generate sequences. This is followed by work on finding rules, as expressed by this learning outcome: 'find the *n*th term, justifying its form by referring to the context in which it was generated'. This looks promising, as the emphasis seems to be on using context to find (and justify) structure, rather than on simply deriving an empirical rule that fits a set of numbers. The learning outcome is exemplified in the Framework document through the use of various figurative patterns, such as the one in Figure 2, in which elements of the pattern are presented in sequence.

In itself, this example is perfectly nice, and it integrates a term-by-term and position-to-term perspective. However, it is less nice if it contributes to a habit of mind where students feel compelled to consider a set of systematic cases and to tabulate the resulting numerical values. The worked example in Figure 3 comes from a current, best-selling Year 8 textbook (Capewell et al., 2003), whose authors have been strongly influenced by the Framework document and examples such as the one in Figure 2.

Notice here that the proposed method seems to be purely numerical and procedural, with no reference to context, that is, to the nature of the given W-pattern. Notice, too, that the pattern itself is not particularly salient and that it is poorly laid out. By contrast, consider the task in Figure 4, taken from a booklet for Year 7 and Year 8 students published more than 20 years earlier (School Mathematics Project [SMP], 1981). Though the given relationship is rather trivial, and one might therefore question why one needs a table, the task is interesting for the fact that the pattern elements are not in any particular order; thus the emphasis is very much on discerning a functional relationship (which is reinforced by the use of arrows in the table and the request for a function 'machine'). Note also that by

Growing matchstick squares



Number of squares	1	2	3	4	…
Number of matchsticks	4	7	10	13	…

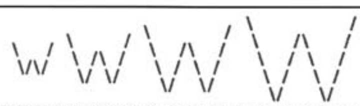
Justify the pattern by explaining that the first square needs 4 matches, then 3 matches for each additional square, or you need 3 matches for every square plus an extra one for the first square.

Figure 2. Exemplification of a Year 7 learning outcome (DfEE, 2001, p. 154).

example

Here are the first four patterns in a sequence:

Find T(n).



The number of lines in the four patterns are:	6	10	14	18
The difference is 4. Compare with $4n$:	4	8	12	16
	2	2	2	2

$T(n) = 4n + 2$.

Figure 3. Worked example from a Year 8 textbook (Capewell et al., 2003, p. 12). Reproduced with kind permission from Oxford University Press © 2003.

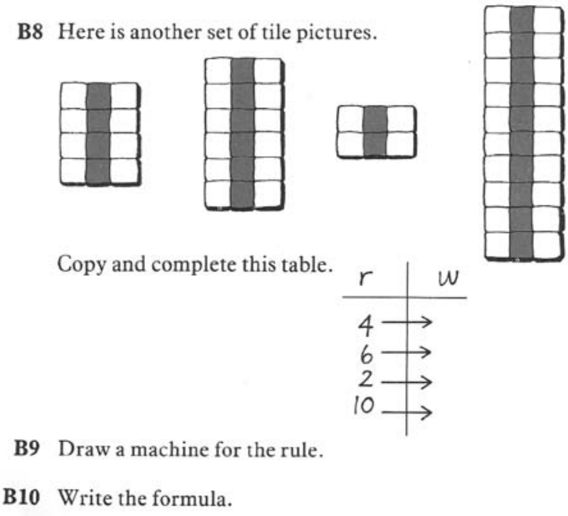


Figure 4. A tile pattern task taken from a Year 7/Year 8 text booklet (SMP, 1981, p. 5) (the grey tiles are coloured red in the actual text, and the letter *r* is used for the number of red tiles). Reproduced with kind permission from The School Mathematics Project © 1981.

using distinct features (red and white tiles) to represent the variables, one can more easily dispense with the use of position to define the independent variable.

Teachers' perception of the power of symbolic algebra

I have been arguing that teachers often seem content to let students form empirical generalizations on the basis of generated numerical data, rather than attempting to discern the structure inherent in a situation directly. Ironically enough, another reason for this would seem to stem from teachers' appreciation of the very power of symbolic algebra.

The teachers involved in the *Proof Materials Project* (Küchemann, 2008) all had a high regard for arguments couched in algebraic terms. One of the tasks we used extensively on the project was this: 'Take any three consecutive numbers and add them together. What do you notice about the totals? Prove that it always happens'.

For the teachers, this was straightforward. They would express the consecutive numbers as, say, n, $n + 1$ and $n + 2$, find the sum, $3n + 3$, transform this into $3(n + 1)$, and hence conclude that the result is a multiple of three (and perhaps also conclude that it is three times the middle number).[1]

In contrast to the teachers, few of the students we worked with used algebraic symbolization. They tended to use numerical examples, and would often generate a systematic set of cases such as:

$$1+2+3=6,\ 2+3+4=9,\ 3+4+5=12,\ 4+5+6=15,\ \text{etc., or}$$
$$1+2+3=6,\ 4+5+6=15,\ 7+8+9=24,\ \text{etc.}$$

leading to the conclusion that the sum 'goes up in threes' or 'goes up in nines', and so on (see Küchemann, 2008, p. 62). However, there were also students who saw that the sum was always three times the middle number and some could explain this in terms of the structure of the task, for example, that ‘the first number is always 1 less than the middle number and the third is always 1 more’. Such a narrative structural argument (which might be likened to ‘rhetorical algebra’) often took the teachers by surprise. I think for some of the teachers, their appreciation of the power of symbolic algebra, can be regarded as an *over*valuing, in that it diminished their appreciation of the benefit of looking for structure and their awareness of the possibilities of doing so. A certain awareness of structure is needed to express the consecutive numbers as n, $n + 1$, $n + 2$; but after this, the symbolization can carry the structure for us and (with perhaps some foresight or luck, and an understanding of syntax) lead us to an expression containing the factor three. Then, if we can read the symbolization, we can conclude that the sum is a multiple of three without necessarily having much feel for why this is so.[2]

Investigational work as assessed coursework

During the 1980s, there was pressure from progressive mathematics educators (such as members of the Association of Teachers of Mathematics, e.g., Ollerton & Watson, 2003), and from some who had over-interpreted the Cockcroft Report’s (1982) reference to ‘investigational work’ (in the much quoted paragraph 243), to include extended investigative tasks in the mathematics curriculum. This, it was hoped, would encourage the development of ‘process’ aspects of mathematics such as conjecturing, explaining and proving. By the end of that decade, to many people’s surprise, the UK government decreed that ‘investigations’, including an extended piece of algebraic coursework, would form part of the national GCSE mathematics examination that students take at age 16 (the statutory school-leaving age in the UK). This seemed like good news to many, but it soon backfired (see Coe & Ruthven, 1994; Hewitt, 1992; Morgan, 1998; Roper, 1999). By the very fact that these tasks were being examined, the approach to them became formulaic, with examination boards providing clear and quite rigid guidance on how they would be marked. In turn, this prescribed how they should be tackled, with an emphasis on generating systematic data, looking for number patterns and making empirical generalizations (with only higher attaining students being expected to justify their empirical rules).[3]

Of the three factors discussed here (the content and exemplification of the English National Curriculum, the overvaluing of algebraic symbolization, and the examination of ‘investigations’), it is perhaps the last of these that has most strongly led to the endorsement by teachers of a number-pattern-spotting approach when it comes to generalization tasks in the mathematics classroom. At the same time, it should be acknowledged that this practice was prevalent before the inclusion of investigations in the GCSE examination.

Thus, consider the task in Figure 5, which comes from a book on problem-solving by Burton (1984). The book contains a rich variety of tasks, but several invite students to draw a table, as in Figure 5, before making a ‘far generalization’ (in this case, about 100 points on a circle). Some students might find the table (and the consideration of cases involving small numbers) helpful, and I would not want to proscribe its use. However, the pattern (or patterns) in the table is trivial, and it is fairly obvious from it that 100 dots will produce $100 - 3$ diagonals, without having to consider the geometry of the situation. Would it not be better (more challenging to students and more informative for the teacher) to start without the table and go straight to the question about 100 dots, and see what happens?

As already mentioned, the practice of number-pattern-spotting is not confined to England or the UK. Consider a study of ‘linear generalizing problems’ undertaken by Stacey (1989), for example. This includes the ‘ladders’ task shown in Figure 6.

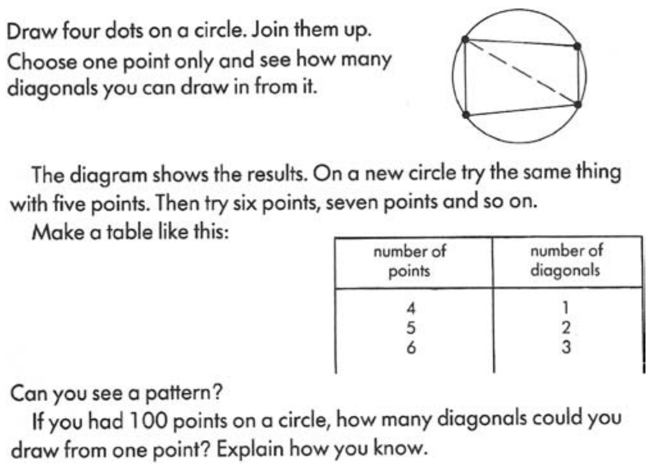


Figure 5. Diagonals task (Burton, 1984, p. 122).
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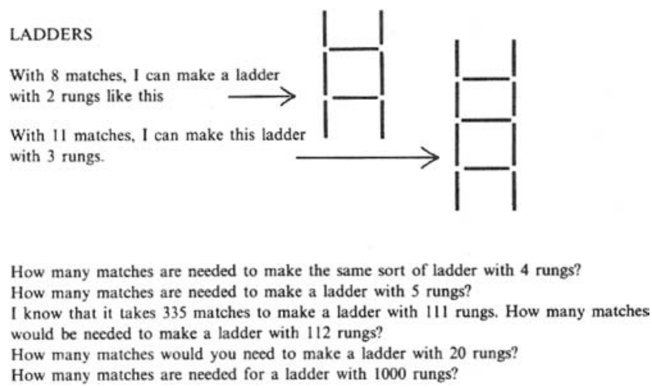


Figure 6. Ladders task (Stacey, 1989, p. 148).
Reproduced with kind permission from Springer © 1989.

It should be acknowledged that this was primarily a study of students' methods and conceptions, rather than of teaching and learning. Thus, the ladders task is not necessarily a teaching task, though it does fit MacGregor and Stacey's (1992) description referred to earlier of common textbook approaches in Australia. The two drawings of ladders are slightly peculiar in that they involve matches of different sizes. That apart, and given students' probable knowledge of ladders, the way the ladders are defined could have been conveyed by a single drawing (though the first drawing, showing the smallest 'practical' ladder, is too special to serve as an effective prototype). So, if this were a classroom task, there might be benefits in presenting just the three-rung ladder, say, and then going straight to the sub-question about 20 or 100 rungs. Stacey describes the four-rung and five-rung sub-questions rather benignly as '"warm-up" exercises to familiarise students with the spatial pattern' (p. 150). It seems likely that these sub-questions will be less neutral in that they may well promote an empirical and term-to-term approach in some students who might otherwise focus on the relationship between number of rungs and total number of matchsticks. Of course, from a research point of view, the sub-questions are perfectly valid, as long as their potential influence on the rest of the task is acknowledged;

and in the event, Stacey's study has thrown up some interesting examples of inappropriate strategies, in particular 'whole-object scaling' (where, e.g., the 100-rung ladder is thought to need five times as many matches as a 20-rung ladder) and other direct proportion methods. It can also be argued that students would (at some stage) benefit from confronting such strategies rather than being shielded from them, and I agree with this. The trouble is, this research, and much related research that has followed (see section below), has almost certainly overstated the ubiquity of such mistaken strategies and the intrinsic difficulty of making far generalizations (i.e., of seeing structure). This point is made by Stacey herself in some of her subsequent work (e.g., Stacey & MacGregor, 2001).

The foregrounding of a number-pattern-spotting approach in current research

A recent issue (Vol. 40, No. 1, 2008) of the research journal *ZDM – The International Journal on Mathematics Education* suggests that the promotion (intended or not) of number-pattern-spotting is as strong as ever in the mathematics education community. The issue contains a short editorial, nine research papers, and a thoughtful (but too polite) commentary by Willi Dörfler (2008).

Six of the nine research papers focus on tasks where the pattern elements are presented sequentially, and it should therefore come as no surprise that they have tended to find a preponderance of number-pattern-spotting approaches. The danger of the near-universal use of such generalization tasks by researchers is illustrated by the authors of one of these six papers, who provide this bold summary:

> As a matter of fact, studies done in different settings (for e.g., countries) and in different contexts (prior to formal instruction in algebra, during and/or after a teaching experiment, etc.) with middle school children have asserted the use of recursion as the entry (and, in some cases, the final) stage in factual generalizing. (Rivera & Becker, 2008, p. 71)

The notion that the use of recursion is a *stage* in the way students generalize seems to me highly dubious – and unproven, despite all the research. Studies exist, albeit relatively few in number, that show that students can adopt a non-recursive approach when patterns are presented in a non-sequential manner.

One such study is by Britt and Irwin (2008) from the same special issue of *ZDM*. They devote part of their paper to discussing a familiar 'borders' task. ('Borders' appears in many guises, e.g., tiles around a swimming pool, or paving stones around a lawn.) They used the context of 'coasters', starting with the single example of a five unit by five unit square coaster, with five counters placed along each side (see Figure 7(a)).

There were two particularly nice features of this task, which was given to a class of 30 13-year-olds working in pairs. One, students were asked to come up with, and illustrate, different counting strategies (as in Figure 7(b)); and, two, they were then asked to apply these to coasters with various non-ordered numbers of counters along each side, for example 100, 47, 139,

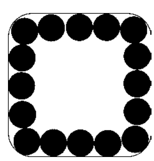


Figure 7(a). Five by five coaster (Britt & Irwin, 2008, p. 44). Reproduced with kind permission from Springer © 2008.

Figure 7(b). Illustration of different counting strategies (Britt & Irwin, 2008, p. 44). Reproduced with kind permission from Springer © 2008.

and to an *n* by *n* coaster. According to Britt and Irwin, all the pairs of students came up with at least one counting strategy, which they represented numerically, using the numbers as quasi-variables, and in terms of *n* (e.g., $2 \times n + 2 \times (n - 2)$ for the first pattern in Figure 7(b)).

Another of the *ZDM* papers that does not focus on sequential examples is by Zazkis, Liljedahl and Chernoff (2008). Their interest is in the role of examples, and they argue that 'the choice of examples that learners are exposed to plays a crucial role in developing their ability to generalize' (p. 131). I strongly agree with this, though their section on 'potential pitfalls in choice of examples' would at first sight seem to counsel against focusing on a single, generic example.

Tasks which prompt a number-pattern-spotting approach and tasks which lend themselves to a generic approach

The third interesting *ZDM* research paper is by Steele (2008), who deliberately sought to reduce students' focus on recursive patterns by discouraging the use of tables. She used two tasks, and in each case, a single example was presented and students were asked to consider another, non-contiguous, case and the general case. In the event, Steele had very limited success, due, I would argue, to an unfortunate choice of tasks. (Interestingly, both tasks had an underlying quadratic structure, though I do not think that in itself was the cause of the students' difficulties.)

In the 'staircase problem' (shown in Figure 8), a four-steps-high staircase is shown and students are asked about a staircase with 18 steps and with *N* steps.

Unfortunately, this task is quite difficult to solve by inspecting a single generic case (unless one happens to see, or know, that two four-steps-high patterns can be put together to form a four by five array of blocks). Instead, it lends itself to a recursive approach (the given staircase requires 1 + 2 + 3 + 4 blocks, and the next staircase will need five extra blocks, the next six extra, the next seven extra, etc). It is therefore not too surprising that many of Steele's small sample of students ($N = 8$) seemed to adopt such an approach. Whether Steele was aware of this characteristic of her task is not made clear.

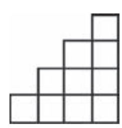
The diagram is the side view of a staircase that is 4 steps high. This staircase is made by stacking 10 blocks as shown. How many blocks would be used to make a staircase that is 18 steps high? If *N* represents the number of steps in a staircase, write an expression to represent how many total blocks are needed to make the staircase	

Figure 8. The staircase problem (Steele, 2008, p. 102). Reproduced with kind permission from Springer © 2008.

The second task that Steele discusses is equivalent to the join-every-dot-to-every-other-dot problem, in her case couched in terms of cities and roads (it is also equivalent to the well-known 'handshakes' problems – every person in a room shakes hands with everyone else). Steele gives the example of four cities, with six direct roads needed to join each city to every other city. The students were asked to find the number of direct roads needed for 14 cities and N cities.

This task can be solved using a generic case: thus for 14 cities, each city will have 14 – 1 roads leading from it; but since each road goes from two cities, the total number of roads is $14 \times (14 - 1) \div 2$, or, for N cities, $N(N - 1)/2$. However, the task also lends itself to a recursive approach, even when contemplating a generic case. Thus, for the 14 cities case, it is easy to adopt this alternative perspective: the first city has 13 roads from it, the second has 12, the third has 11, and so forth, leading to 13 + 12 + . . . + 1 + 0 for the total number of roads, or $N - 1 + N - 2 + \ldots + (N - (N - 1)) + (N - N)$ in the case of N cities. (This perspective is likely to come to the fore when one *draws* the cities and roads; perhaps not so likely when one *counts* them.)

In the event, four of Steele's eight students used this latter approach, and three used a more standard recursive approach by creating a table of values. Only one student went directly to a closed (functional) rule using a generic approach, in this case based on 'diagonal' roads (of which there are $N - 3$ from each city) and roads on the 'perimeter' (of which there are N), leading to the formula $(N/2) \times (N - 3) + N$, which of course is equivalent to $N(N - 1)/2$.

For both of Steele's tasks, it is possible (albeit relatively difficult) to see the structure directly and in a non-recursive way, from a single, generic example. I now want to consider a task, shown in Figure 9, where a non-recursive approach would seem to be extremely difficult, and perhaps impossible.

For the cities and roads task, it was possible to come up with a statement like 'each of the 14 cities has 13 roads leading from it', which can take one direct to a general, closed rule. In the current task, a similar statement can be made about the lines and regions, but this time it does not seem to be fruitful. Thus, for the four-line example shown in Figure 9, we can state that 'every line borders eight regions' (and we can explain why: each time a given line cuts one of the other three lines, and also when it exits the circle, it cuts a region in two). However, this does not seem to get us any further: whereas we could say each road is joined to two points so that we need to divide the 'total' number of roads by two, there is no such obvious way of sharing the eight regions associated with each line; some regions are bounded by three lines (or two lines and an arc), some by four. Put another way, how might one map 'four lines and eight regions per line' onto the known total of 11 regions?!

It turns out that a more productive approach here is to work recursively, by describing what happens when an *extra* line is added: thus, adding a fifth line produces five extra

If n straight lines all intersect each other (inside a circle), how many regions do they make (inside the circle)? Explain why.

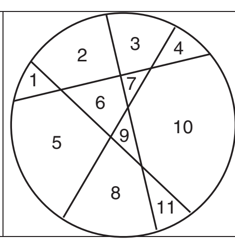


Figure 9. Lines and regions task.

regions (one for each of the four lines it meets and one more when it ‘exits’ the circle). So we end up with a recursive rule: n lines produce $1 + 1 + 2 + \ldots + n$ regions. This, in turn, can be transformed into the closed rule $1 + n(n + 1) \div 2$ (as long as we have the necessary insight or expertise).

The example of this task, together with Steele’s paper, would seem to provide a strong argument against the generic approach that I am advocating. However, I think the situation is more subtle. First, I would like to repeat that I am not advocating that we banish a term-to-term (or recursive or ‘incremental’ or ‘dynamic’) approach. Sometimes this is likely to be the only fruitful course open to us. It can also be useful for examining the properties of geometric shapes. And when it comes to the graphs of functions, increments are closely allied to the notion of *gradient*. Further, students need to be able to unpick the inappropriate strategies that a term-to-term approach can engender and to see how term-to-term and position-to-term approaches can complement each other.

What I want to argue is that a generic approach, where it is viable, can focus the attention of students (and teachers and researchers) directly on the search for structure, from which they are often distracted by current classroom practices. We want students to ‘look at the problem’, as Polya put it (see Walter, 2003, Footnote 1), rather than immediately generate data and draw a table, which can so easily turn into displacement activity.

It is worth pointing out, too, that when a purely numerical approach is adopted, it is possible to find any number of rules that fit a given set of ordered pairs (see, e.g., Hewitt, 2008). A classic illustration of the dangers of such an approach is the ‘pancakes’ task shown in Figure 10. It turns out that the number of regions for six dots is not 32, as the number pattern in the table so strongly suggests, but 31 (see Hart, 2007, for a nice solution).

Many tasks involving figurative patterns, including most of those discussed in the special issue of *ZDM*, can be tackled generically. Figure 11 shows how a typical task of this sort (which happens to have a quadratic structure) can be presented generically, that is, by showing just a single element of the pattern, and with no suggestion that students should investigate the ‘next’ pattern elements in order to make a generalization. An interesting approach to a similar task is described by Wall (2001), who asked students to form

n points are drawn on a circle. A line is drawn from each point to every other point. What is the maximum number of regions that can be formed inside the circle? Why?

Number of dots	Maximum number of regions
1	1
2	2
3	4
4	8
5	16
6	?

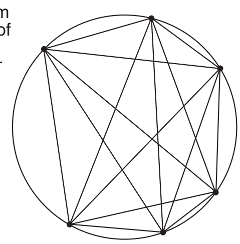


Figure 10. Pancakes task.

This is a 3 by 3 square of matchsticks.

How many matchsticks are needed for

- a 20 by 20 square
- an n by n square?

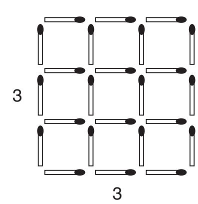


Figure 11. Square grid made of matchsticks task.

a mental image of the three by three array and discouraged them from drawing a series of arrays. The task, again presented generically, was also used by Hershkowitz, Arcavi and Bruckheimer (2001) with a variety of teachers from different countries. They found that many teachers could see the structure directly from the example given, but others adopted a number-pattern-spotting approach. It seems that 'some of the teachers were unaware of or unable to appreciate that a general solution and its justification can be produced entirely visually' (p. 263).

It would be nice to be able to identify the characteristics of tasks that lend themselves to a generic approach. Unfortunately, I have not succeeded in doing this, but others may have. It helps if the independent variable is a directly discernable feature of a pattern element (e.g., the number of squares or tables in a row), rather than having to be determined by the position of an element in a sequence; it helps, too, if the variables are distinct (e.g., squares and matchsticks, tables and chairs, cities and roads). However, while these are desirable characteristics, they are neither necessary nor sufficient!

A small-scale classroom intervention that used a generic approach

One of the teachers on the Proof Materials Project (Küchemann, 2008) adapted this task (Figure 11) for use with her 'low-attaining' Year 7 class (Grade 6, age 11–12).[4]

Here I report on the work done over the course of two lessons by two fairly typical students, Lyn and Lyle. I will argue that, having been given the opportunity to think about structure, these students (as with most others in the class) show clear evidence of being able to do so, even if their notions of generalizing in mathematics are not well formed.

The teacher was experienced and skilled. She did not normally use a generic approach when using figurative patterns but was interested in seeing how this might work. She was also comfortable about taking risks and was prepared to give students time to explore ideas.

In the first lesson, each student was given a sheet of A5 paper on which was printed two three by three matchstick square grids. The students were asked to find efficient ways of counting the matches and to show this by colouring groups of matches. The potential value of looking at counting strategies has already been mentioned (Britt & Irving, 2008; see also Zazkis et al., 2008). Hewitt puts it like this: 'To be able to count requires a way of counting, a way of structuring and organising counting. To be able to count requires you to work algebraically' (1998, p. 20).

Lyn's and Lyle's drawings are shown in Figures 12(a) and 12(b). In Lyn's first drawing, she has structured the grid into (green) rows and (brown) columns. The structure in her second drawing is more complex and can be thought of as an outer (green) border, two (brown) crosses and two (red) L-shapes. This structure is probably not easily generalizable, in contrast to the first one.

Lyle's first structuring (Figure 12(b)) consists of rows (blue) and columns (yellow), like Lyn's. His second structuring can be viewed as an outer (blue–grey) border and two inner (yellow) rows and (blue) columns. This is again more complex than the first structure but could be generalized quite easily.

These and other students' drawings suggest that most of the students in this low-attaining set could find effective ways of structuring the matchstick array. Of course, one must be careful not to read too much into these drawings. For example, consider Lyn's first drawing, of green rows and brown columns: it seems to me quite possible that she would have been able to express this numerically, as something like 'four rows of three and four columns of three'. However, while some more experienced students would be able to express this

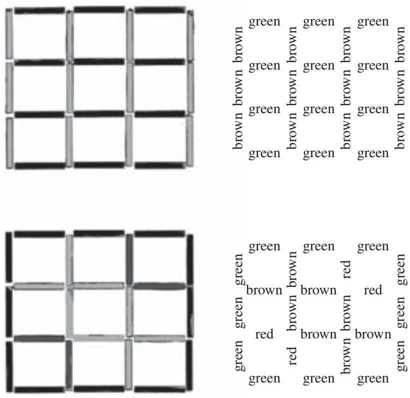


Figure 12(a). Lyn’s drawing.

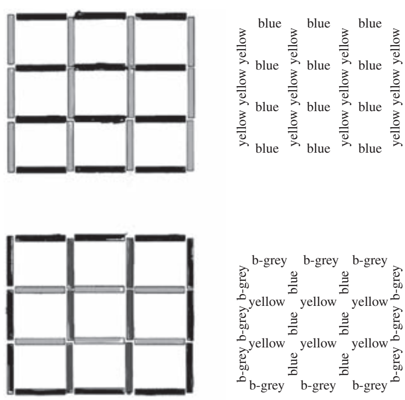


Figure 12(b). Lyle’s drawing.

in all its generality (e.g., as $2 \times n \times (n + 1)$ for an n by n matchstick square), it is not necessarily the case that the numbers three and four are being seen generically by Lyn, that is, with the three corresponding to the width (and height) of the matchstick square and the four indicating the need for an ‘extra’ row (and column). It is a pity that the teacher did not ask the students to represent their counting methods as numerical expressions, alongside their drawings. It would also have been interesting to see how the students might, immediately after this, enumerate a 10 by 10 matchstick square, especially if asked to do so without drawing.

The follow-up lesson did not probe these questions directly (in particular, whether students could generalize one of their own methods if asked to do so). However, it did throw some interesting light on students’ spontaneous generalizing strategies. In particular, the follow-up lesson suggests that these students could discern general structural features of a figurative pattern, but also that they did not yet have the experience to discriminate between effective (in the sense of efficient and/or easily generalizable) and less effective structures. In this second lesson students were asked to predict and then find a way of determining the number of matchsticks in a four by four, a five by five, and a 10 by 10 array. Here, the teacher was not sticking rigorously to a generic approach – however, the focus was very much on the patterns, which the students were asked to draw, not on numbers abstracted from them.

The extract in Figure 13 shows the first page of Lyn's work in this second lesson. As can be seen, her predictions are not correct: 34 rather than 40 for the four by four grid; 44 rather than 60 for the five by five; 84 rather than 220 for the 10 by 10 grid. However, she is honest and perhaps confident enough not to blank out her wrong predictions. Her annotations indicate that she has a nice, systematic method for evaluating the four by four grid, although it does not match either of the ways in which she structured the three by three grid in Lesson 1. She seems to have counted the four lines of four matchsticks around the outside, the three rows of four, and the three columns of four. She now does show some consistency in that she uses the same method for the five by five grid and does so successfully (see her Page 2, shown in Figure 14). Moreover she applies the method to the 10 by 10 grid and at a greater level of abstraction in that she does this without a diagram. This is impressive, though she makes the mistake of thinking there are 10 inner rows and columns rather than nine.

The structures used by Lyle in Lesson 2 differ from those in Lesson 1, as was the case with Lyn. Furthermore, he is inconsistent within Lesson 2. His method for the four by four grid is unsystematic, though he comes close to finding the right number of matches (see Figure 15). However, he does use a systematic structuring for the five by five grid and copes with it very well, despite its complexity (his only mistake is to evaluate 15×3 as 48 rather than 45).

Lyle's response for the 10 by 10 grid is particularly interesting. He draws only a schematic diagram and it would appear that this time he has come up with a somewhat

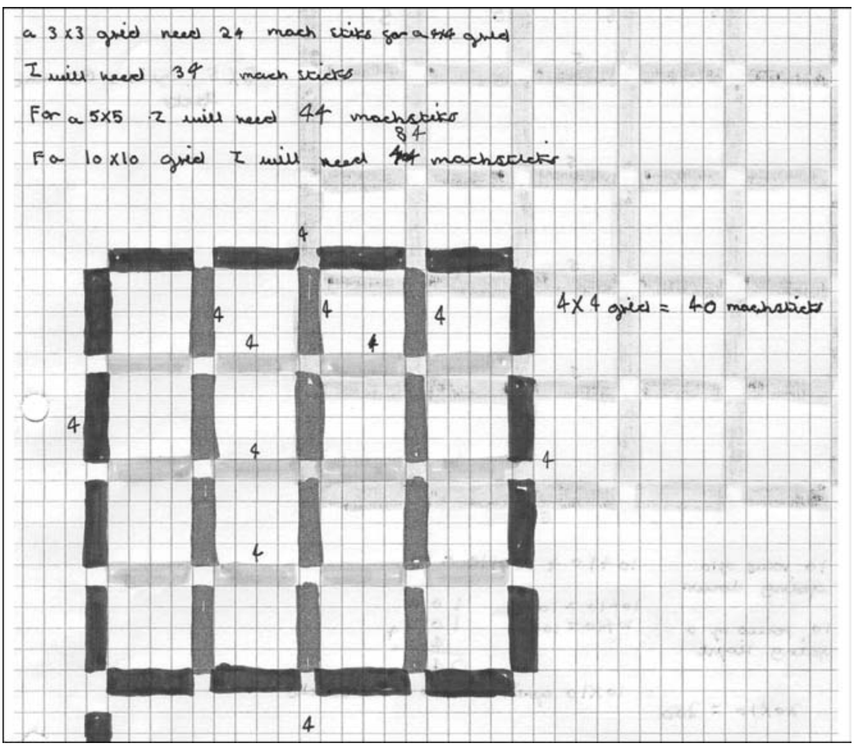


Figure 13. Lyn, Lesson 2, Page 1.

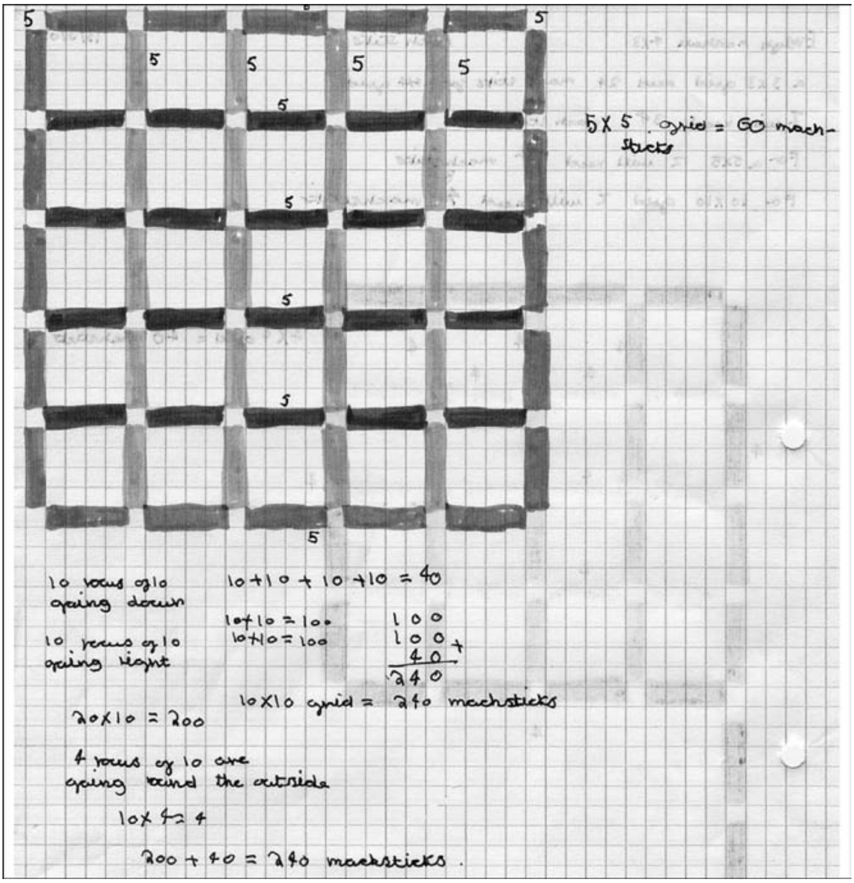


Figure 14. Lyn, Lesson 2, Page 2.

simpler and more easily generalizable structure, namely, a border of matchsticks with rows and columns inside. This is similar to the structure used by Lyn for the 10 by 10 grid (but whereas she mistakenly thought there were 10 inner rows and columns rather than nine, he thinks there are eight). As with Lyn, this attempt to structure the 10 by 10 grid is impressive and would seem to indicate the beginnings of a fruitful generalization. Moreover, it seems likely that this was brought on by the teacher's careful choice of a grid size, which is relatively large and therefore difficult and time-consuming to draw properly. (It is interesting to consider whether a 20 by 20 grid, say, might have been even better!)

Looking back at these two lessons, it may well be that there was a tension for these inexperienced students between wanting them to produce alternative ways of structuring a given situation (Lesson 1) and wanting them to generalize (Lesson 2). Perhaps the students should have been asked to choose one of their methods from Lesson 1 and to try to apply it in Lesson 2. Of course, finding alternatives can be a valuable activity and it can lead to challenging work on showing that different structurings are equivalent, which is perhaps especially instructive once the structures have been expressed algebraically.

It has not been possible to follow the progress of these students further, so we do not know what lasting effects these two lessons, and further lessons of this sort, might have had. However, the very fact that these students were able to discern structure to some

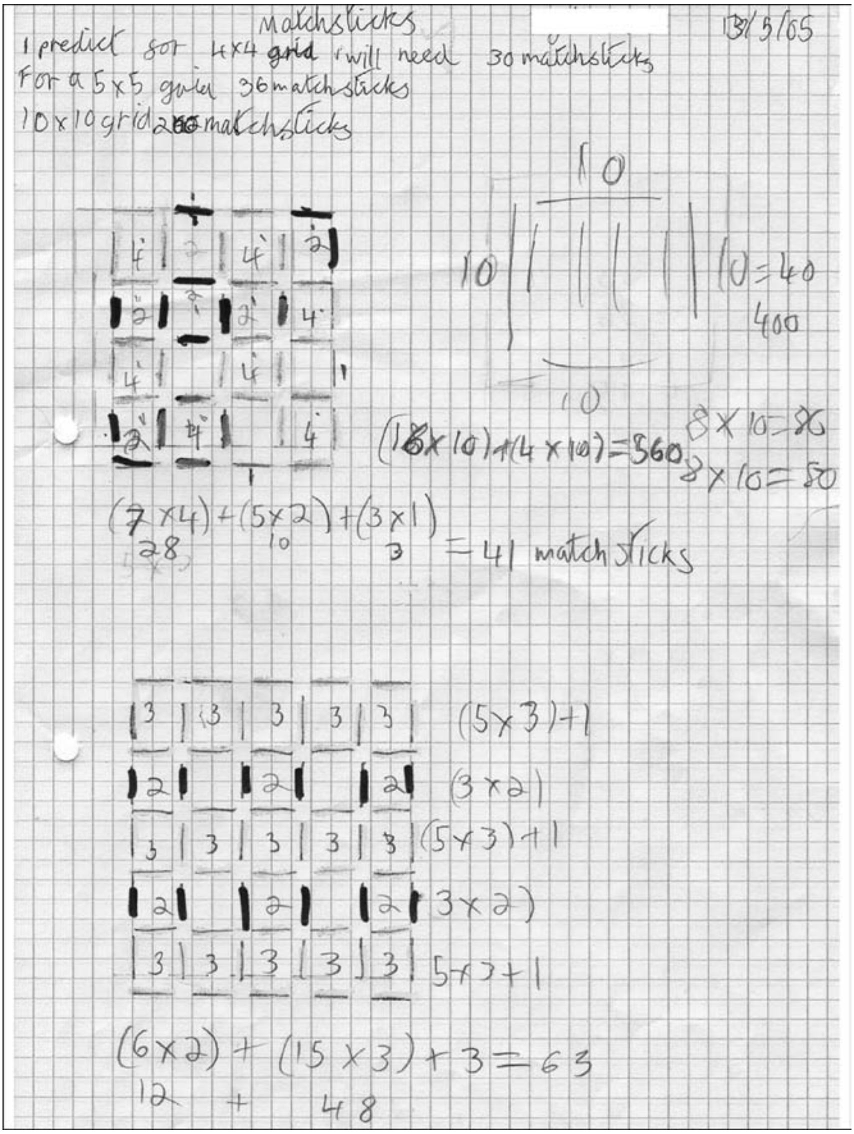


Figure 15. Lyle, Lesson 2.

extent suggests that a generic approach offers a viable alternative to the empirical approach promoted by many teachers.

Conclusion

There is evidence from a variety of situations and sources that students have a tendency to argue at an empirical level rather than in terms of structure (e.g., Healy & Hoyles, 2000). However, in part, and perhaps in large measure, this must stem from students' lack of experience of looking for structure and a lack of encouragement to do so.

In this paper I have referred to a small number of studies and given some direct evidence to suggest that school students' struggle to form structural generalizations is overstated. I have also argued that the situation is compounded by presenting generalization tasks in the form of sequential elements rather than generically. In the light of a study with prospective elementary and middle school teachers in the US, Rivera and Becker (2005) recommend that 'teachers need to give their students activities and problem situations that de-emphasize the numerical and emphasize a figural understanding of generalization' (p. 202). Though they illustrate their example with a task that involves a pattern sequence, I agree that we should do more to focus the attention of teachers (and hence their students) on structure, and I suggest that a generic approach involving figural patterns would help in this.

My experience of working with mathematics teachers in English secondary schools suggests that this is not always easy to implement. Though this alternative approach may be cognitively more demanding, I think it primarily requires a change of outlook, of habit of mind – and an ability to overcome the various influences discussed earlier in this paper that seem to promote a number-pattern-spotting approach. As with some of Rivera and Becker's prospective teachers, some practising teachers may not be in the habit of looking for mathematical structure, and will thus need experience of thinking in this way. Understandably, too, teachers can be attracted by the air of order and industry engendered by students who are busily compiling lengthy tables of values – students who feel satisfied by their industry, all the more so for having avoided the stress of mathematical thought. By contrast, the direct focus on structure inherent in a generic approach involves uncertainty and risk, and the disruption of this industrious idyll. Thus some teachers need experience of working in this way in the classroom, to discover it can also involve challenge and excitement, and the pleasure of engaging with mathematical ideas.

Notes

1. A more elegant but less common way of expressing the consecutive numbers, and one that shows a greater awareness of structure, is to use $n - 1$, n and $n + 1$, leading directly to $3n$ for their sum; see Arcavi (1994).
2. A corresponding difficulty exists for students, which I will not develop here: we want them to learn to use symbolic algebraic to express structure, but the very symbolization may well distance them from seeing it. The argument here is that teachers, especially some who appreciate the power of algebra, may well not focus on or promote structural generalizations expressed in narrative form. However, this does not mean that they do not appreciate such generalizations when they come across them. See, for example, Küchemann and Hoyles (2003).
3. Since 2009, through ministerial decree, coursework is no longer part of the mathematics GCSE examination, though not for mathematical reasons but because of the perceived dangers of cheating, by teachers, parents and students, made more acute by the availability of model answers on the Internet.
4. As is common in England, the mathematics class was formed on the basis of 'attainment', and this class was deemed to be the third-lowest attaining mathematics class out of the four Year 7 classes in the school.

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